OPTIMAL EQUITABLE SYMBOL WEIGHT CODES FOR POWER LINE COMMUNICATIONS

Y. M. CHEE †¶, H. M. KIAH †¶, A. C. H. LING ‡, AND C. WANG §¶

Abstract. The use of multiple frequency shift keying modulation with permutation codes addresses the problem of permanent narrowband noise disturbance in a power line communications (PLC) system. Equitable symbol weight codes was recently demonstrated to optimize the performance against narrowband noise in a general coded modulation scheme. This paper establishes the first infinite families of optimal equitable symbol weight codes with code lengths greater than alphabet size and whose narrowband noise error-correcting capability to code length ratios do not diminish to zero as the length grows. These families of codes meet the generalized Plotkin bound. The construction method introduced is combinatorial and reveals interesting interplay with an extension of the concept of generalized balanced tournament designs from combinatorial design theory.

Key words. multiple frequency shift key modulation, power line communications, equitable symbol weight codes, generalized balanced tournament designs, recursive constructions

AMS subject classifications. 94B99, 94C30, 05B30

1. Introduction. Power line communications (PLC) is a technology that enables data transmission over high voltage electric power lines. Started in the 1950s in the form of ripple control for load and tariff management in power distribution, this low bandwidth one-way communication system evolved to a two-way communication system in the 1980s. With the emergence of the Internet in the 1990s, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides "last mile" connectivity to premises and capillarity within premises. Recently, there has been a renewed interest in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [21, 24, 33, 43]).

However, power lines present a difficult communications environment and overcoming permanent narrowband disturbance has remained a challenging problem [3,35, 39]. Vinck [39] addressed this problem by showing that multiple frequency shift keying (MFSK) modulation, in conjunction with the use of a permutation code having minimum (Hamming) distance d, is able to correct up to $d-1$ errors due to narrowband noise. Since then, more general codes such as constant-composition codes, frequency permutation arrays, and injection codes (see [7,12–20,23,25–27,32,34]) have been considered as possible replacements for permutation codes in PLC. Versfeld *et al.* [37,38] later introduced the notion of 'same-symbol weight' (henceforth, termed as *symbol weight*) of a code as a measure of the capability of a code in dealing with narrowband noise. They also showed empirically that low symbol weight cosets of Reed-Solomon

†Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Singapore (ymchee@ntu.edu.sg; kiah0001@ntu.edu.sg).

‡Department of Computer Science, University of Vermont, Burlington, VT 05405, USA (email: aling@emba.uvm.edu).

§School of Science, Jiangnan University, Wuxi 214122, China, (email: chengmin_wang09@yahoo.com.cn). Research of C. Wang is supported in part by NSFC under Grant No.11271280.

¶Research of Y. M. Chee, H. M. Kiah and C. Wang is supported in part by the National Research Foundation of Singapore under Research Grant NRF-CRP2-2007-03. C. Wang is also supported in part by NSFC under Grant No.10801064 and 11271280.

codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise.

Unfortunately, symbol weight alone is not sufficient to capture the performance of a code in dealing with permanent narrowband noise. Chee *et al.* [9] extend the analysis of Vinck's coded modulation scheme based on permutation codes (see [39], [2, Subsection 5.2.4]) to general codes. They introduce an additional new parameter that more precisely captures a code's performance against permanent narrowband noise. This parameter is related to *symbol equity*, the uniformity of frequencies of symbols in each codeword. Codes designed taking into account this new parameter, or *equitable symbol weight codes*, are shown to perform better than general ones.

Relatively little is known about optimal equitable symbol weight codes, other than those that correspond to injection codes (which include the class of permutation codes) and frequency permutation arrays. In particular, only six infinite families of optimal equitable symbol weight codes with code length greater than alphabet size are known. These have all been constructed by Ding and Yin [18], and Huczynska and Mullen [26] as frequency permutation arrays and they meet the Plotkin bound. One drawback with the code parameters of these families is that the narrowband noise error-correcting capability to length ratio diminishes as its length grows.

In this paper, we construct the first infinite families of optimal equitable symbol weight codes, whose code lengths are larger than alphabet size and whose narrowband noise error-correcting capability to length ratios tend to a positive constant as code length grows. These families of codes all attain the generalized Plotkin bound. Our results are based on the construction of equivalent combinatorial objects called generalized balanced tournament packings.

Owing to space constraints, we omit listing the words of certain small codes and blocks of certain combinatorial objects. We refer the interested reader to the second author's website [10] for complete descriptions. Parts of the paper have been presented in [8].

2. Preliminaries.

2.1. Notations. For positive integer m and prime power q, denote the ring $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m$ and the finite field of q elements by $\mathbb{F}_q$. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Let $[m]$ denote the set $\{1, 2, \ldots, m\}$. We use angled brackets ($\langle$ and $\rangle$) for multisets. Disjoint set union is depicted using $\sqcup$. For sets A and B, an element $(a, b) \in A \times B$ is sometimes written as a_b for succinctness.

If T is a multiset of integers and m is an integer, then $T + m$ denotes the multiset $\langle t + m : t \in T\rangle$, mT denotes the multiset $\langle m \cdot t : t \in T\rangle$, and $T \bmod m$ denotes the multiset $\langle t \bmod m : t \in T\rangle$. Let Γ be an additive abelian group, $\gamma \in \Gamma$, and A be a set. If T is a multiset of elements from $\Gamma \times A$, then $T + \gamma$ denotes the set $\langle (t + \gamma)_a : t_a \in T\rangle$. Furthermore, if $\Gamma = \mathbb{Z}_m$ and m' is an integer such that $m \equiv 0 \bmod m'$, then $T \bmod m'$ denotes the set $\langle (t \bmod m')_a : t_a \in T\rangle$. Also, we embed $\mathbb{Z}_{m'}$ into $\mathbb{Z}_m$ by $i + m'\mathbb{Z} \mapsto i + m\mathbb{Z}$, so that $T + j$ makes sense for $j \in \mathbb{Z}_{m'}$. Note that a set is also a multiset and hence our definitions for multisets apply to sets.

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{A})$, where X is a finite set of *points* and $\mathcal{A} \subseteq 2^X$. Elements of $\mathcal{A}$ are called *blocks*. The *order* of $\mathfrak{S}$ is the number of points in X, and the *size* of $\mathfrak{S}$ is the number of blocks in $\mathcal{A}$. Let K be a set of nonnegative integers. The set system $(X, \mathcal{A})$ is said to be K-*uniform* if $|A| \in K$ for all $A \in \mathcal{A}$.

2.2. Equitable Symbol Weight Codes. Let Σ be a set of q *symbols*. A q-*ary code* of *length* n over the *alphabet* Σ is a subset $\mathcal{C} \subseteq \Sigma^n$. Elements of $\mathcal{C}$ are called

codewords. The *size* of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. For $i \in [n]$, the ith coordinate of a codeword $\mathsf{u} \in \mathcal{C}$ is denoted u_i, so that $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_n)$.

Denote the *frequency* of symbol $\sigma \in \Sigma$ in codeword $\mathsf{u} \in \Sigma^n$ by $w_\sigma(\mathsf{u})$, that is, $w_\sigma(\mathsf{u}) = |\{\mathsf{u}_i = \sigma : i \in [n]\}|$. An element $\mathsf{u} \in \Sigma^n$ is said to have *equitable symbol weight* if $w_\sigma(\mathsf{u}) \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$ for any $\sigma \in \Sigma$. If all the codewords of $\mathcal{C}$ have equitable symbol weight, then the code $\mathcal{C}$ is called an *equitable symbol weight code*.

Consider the usual Hamming distance defined on codewords and codes and let d denote the minimum distance of a code $\mathcal{C}$. In addition, consider the following parameter.

Definition 2.1. *Let $\mathcal{C}$ be a q-ary code with minimum distance d. The* narrowband noise error-correcting capability *of $\mathcal{C}$ is*

$$c(\mathcal{C}) = \min\{e : E_{\mathcal{C}}(e) \geq d\},$$

where $E_{\mathcal{C}}$ is a function $E_{\mathcal{C}} : [q] \to [n]$, given by

$$E_{\mathcal{C}}(e) = \max_{\substack{\Gamma \subseteq \Sigma \\ |\Gamma| = e}} \max_{\mathsf{c} \in \mathcal{C}} \left\{ \sum_{\sigma \in \Gamma} w_\sigma(\mathsf{c}) \right\}.$$

Chee *et al.* [9] established that a code $\mathcal{C}$ can correct up to $c(\mathcal{C}) - 1$ narrowband noise errors and demonstrated that an equitable symbol weight code maximizes the quantity $c(\mathcal{C})$, for fixed n, d and q. Henceforth, only equitable symbol weight codes are considered. A q-ary equitable symbol weight code of length n having minimum distance d is denoted $(n, d)_q$-*equitable symbol weight code*. Denote the maximum size of an $(n, d)_q$-equitable symbol weight code by $A_q^{ESW}(n, d)$. Any $(n, d)_q$-equitable symbol weight code of size $A_q^{ESW}(n, d)$ is said to be *optimal*.

Taken as a q-ary code of length n, an optimal $(n, d)_q$-equitable symbol weight code satisfies the generalised Plotkin bound [28, Ch.2, Theorem 2.82, Corollary 2.84, Theorem 2.86].

Theorem 2.2 (Generalised Plotkin Bound). *If there is a $(n, d)_q$-equitable symbol weight code $\mathcal{C}$ of size M, then*

$$\binom{M}{2} d \leq n \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} M_i M_j, \tag{2.1}$$

where $M_i = \lfloor (M + i)/q \rfloor$. If q divides M and $\binom{M}{2} d = n \binom{q}{2} (M/q)^2$, then $\mathcal{C}$ is optimal.

In the rest of this paper, equitable symbol weight codes whose sizes attain the generalised Plotkin bound are constructed. In particular, the following is established.

Theorem 2.3. *The following holds.*

(i)

$$A_q^{ESW}(2q-1, 2q-2) = \begin{cases} 3, & q = 2, \\ 2q, & q \geq 3, \end{cases}$$

(ii)

$$A_q^{ESW}(2q-3, 2q-4) = \begin{cases} 6q - 12, & q = 3, 4, \\ 14, & q = 5, 6, \\ 2q + 1, & q \geq 7, \textit{except possibly } q \in \{12, 13\} \end{cases}$$

(iii)

$$A_q^{ESW}(3q-1, 3q-3) = \begin{cases} 4, & q = 2, \\ 3q, & q \geq 3, \end{cases}$$

(iv)

$$A_q^{ESW}(4q-1, 4q-4) = \begin{cases} 4q-1, & q = 2, 3, \\ 4q, & q \geq 4, \end{cases}$$

(v)

$$A_q^{ESW}\left(\frac{3q-1}{2}, \frac{3q-3}{2}\right) = \begin{cases} 4q-6, & q = 3, 5, \\ 3q, & q \geq 7 \textit{ is odd}, \end{cases}$$

(vi) $A_q^{ESW}(\frac{\lambda(kq-1)}{k-1}, \frac{\lambda k(q-1)}{k-1}) = kq$ *where* $k \geq 4$, $\lambda \in [k-1]$ *and* $q \equiv 1 \bmod k(k-1)$ *is a sufficiently large prime power.*

Observe that any equitable symbol weight code $\mathcal{C}$ with the above parameters must have $c(\mathcal{C}) = q-1$. In Table 2.1, we verify that $c(\mathcal{C})/n$ tends to a positive constant as q grows. In the same table, we compare with known families of optimal $(n, d)_q$-equitable symbol weight codes.

In particular, only six infinite nontrivial families of optimal codes with $n > q$ are known. However, code parameters for these six families are such that their relative narrowband noise error-correcting capability to length ratios diminish to zero as q grows. This is undesirable for narrowband noise correction for PLC. Hence, Theorem 2.3 provides the first infinite families of optimal equitable symbol weight codes with code lengths are larger than alphabet size and whose relative narrowband noise capability to length ratios tend to a positive constant as length grows.

Our construction of optimal equitable symbol weight codes employs tools from combinatorial design theory.

Table 2.1

Infinite families of optimal $(n,d)_q$ equitable symbol weight codes

$(n,d)_q$-equitable symbol weight code $\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$c(\mathcal{C})$	$\lim_{q\to\infty} c(\mathcal{C})/n$	Remarks
$(n,n)_q$ for $q \geq 2$	q	$\min\{n,q\}$	–	easy
$(3,2)_q$ for $q \geq 3$	$q(q-1)$	2	0	injection code [20]
$(4,2)_q$ for $q \geq 4$, $q \neq 7$	$q(q-1)(q-2)$	2	0	injection code [20]
$(n,1)_q$ for $n < q$	$q(q-1)\cdots(q-n+1)$	1	$1/n$	injection code, easy
$(q,2)_q$ for $q \geq 2$	$q!$	2	0	injection code, easy
$(q,3)_q$ for $q \geq 3$	$q!/2$	3	0	injection code, easy
$(q,q-1)_q$ for prime powers q	$q(q-1)$	$q-1$	1	injection code [15]
$(n,n-1)_q$ for q sufficiently large and $n \leq q$	$q(q-1)$	$n-1$	$1-1/n$	injection code [20]
$(q,q-2)_q$ for prime powers $q-1$	$q(q-1)(q-2)$	$q-2$	1	injection code [22]
$\left(q(q+1),q^2\right)_q$ for prime powers q	q^2	q	0	frequency permutation array [17]
$\left(\frac{q(kq^2-1)}{k-1}, \frac{kq^2(q-1)}{k-1}\right)_q$ for prime powers q, $2 \leq k \leq 5$, $(k,q) \neq (5,9)$	kq^2	q	0	frequency permutation array [18]
$\left(\frac{\mu q^{s-t}(q^{2s-t}-1)}{q^t-1}, \frac{\mu q^{2s-t}(q^{s-t}-1)}{q^t-1}\right)_{q^{s-t}}$ for prime powers q, $1 \leq t < s$, $\mu = \prod_{i=1}^{t-1} \frac{q^{s-i}-1}{q^i-1}$	q^{2s-t}	q^{s-t}	0	frequency permutation array [18]
$(q^s(q^{2s+c}-1), q^{2s+c}(q^s-1))_{q^s}$, for prime powers q, and $s,c \geq 1$	q^{2s+c}	q^s	0	frequency permutation array [18]
$(\binom{kq}{k}, \frac{kq-k}{kq-1}\binom{kq}{k})_q$ for $q,k \geq 1$	kq	$q-1$	0	frequency permutation array [26]
$(2q^2-q, 2q^2-2q)_q$ for even q, $q \notin \{2,6\}$	$2q$	q	0	frequency permutation array [26]
$(2q-1,2q-2)_q$ for $q \geq 3$	$2q$	$q-1$	$1/2$	Theorem 2.3
$(2q-3,2q-4)_q$ for $q \geq 14$	$2q+1$	$q-2$	$1/2$	Theorem 2.3
$(3q-1,3q-3)_q$ for $q \geq 3$	$3q$	$q-1$	$1/3$	Theorem 2.3
$(4q-1,4q-3)_q$ for $q \geq 4$	$4q$	$q-1$	$1/4$	Theorem 2.3
$\left(\frac{3q-1}{2}, \frac{3q-3}{2}\right)_q$ for $q \geq 7$ and q odd	$3q$	$q-1$	$2/3$	Theorem 2.3
$\left(\frac{\lambda(kq-1)}{k-1}, \frac{\lambda k(q-1)}{k-1}\right)_q$ for $k \geq 4$, $\lambda \in [k-1]$, q is a sufficiently large prime power with $q \equiv 1 \bmod k(k-1)$	kq	$q-1$	$(k-1)/(\lambda k)$	Theorem 2.3

3. Constructions of Equitable Symbol Weight Codes from Combinatorial Objects. This section introduces the necessary concepts and establishes their connections to equitable symbol weight codes.

However, we first determine $A_q^{ESW}(n,d)$ for small values of n, q and d. With the exception of $A_6^{ESW}(9,8)$, an exhaustive computer search established the following values of $A_q^{ESW}(n,d)$. For $A_6^{ESW}(9,8)$, a $(9,8)_6$-equitable symbol weight code of size 14 was found via computer search. Since a $(9,8)_6$-equitable symbol weight code of size 15 cannot exist by the generalized Plotkin bound, it follows that $A_6^{ESW}(9,8) = 14$. We record the results of the computations in the following proposition and the corresponding optimal codes can be found at [10].

Proposition 3.1. *The following holds:*

$$\begin{array}{lll} A_2^{ESW}(3,2) = 3 & A_2^{ESW}(5,3) = 4 & A_2^{ESW}(7,4) = 7 \\ A_3^{ESW}(3,2) = 6 & A_3^{ESW}(4,3) = 6 & A_3^{ESW}(11,8) = 11 \\ A_4^{ESW}(5,4) = 12 & A_5^{ESW}(7,6) = 14 & A_6^{ESW}(9,8) = 14. \end{array}$$

The rest of the paper establishes the remaining values in Theorem 2.3.

3.1. Equitable Symbol Weight Codes and Generalized Balanced Tournament Packings. Let λ, v be positive integers and K be a set of nonnegative integers. A (v,K,λ)-*packing* is a K-uniform set system of order v such that every pair of distinct points is contained in at most λ blocks. A *parallel class* (or *resolution class*) of a packing is a subset of the blocks that partitions the set of points X. If the set of blocks can be partitioned into parallel classes, then the packing is *resolvable*, and denoted by $\mathrm{RP}(v,K,\lambda)$. An $\mathrm{RP}(v,K,\lambda)$ is called a *maximum resolvable packing*, denoted by $\mathrm{MRP}(v,K,\lambda)$, if it contains maximum possible number of parallel classes.

Furthermore, an $\mathrm{MRP}(v,\{k\},\lambda)$ is called a *resolvable* $(v,\{k\},\lambda)$-*balanced incomplete block design*, or $\mathrm{RBIBD}(v,k,\lambda)$ in short, if every pair of distinct points is contained in exactly λ blocks. A simple computation gives the size of an $\mathrm{RBIBD}(v,k,\lambda)$ to be $\frac{\lambda v(v-1)}{k(k-1)}$.

Definition 3.2. *Let $(X,\mathcal{A})$ be an RP(v,K,λ) with n parallel classes. Then $(X,\mathcal{A})$ is called a* generalized balanced tournament packing *if the blocks of $\mathcal{A}$ are arranged into an $m\times n$ array satisfying the following conditions:*

(i) every point in X is contained in exactly one cell of each column,

(ii) every point in X is contained in either $\lceil n/m\rceil$ or $\lfloor n/m\rfloor$ cells of each row.

We denote such a GBTP by GBTP$_\lambda(K;v,m\times n)$.

Unless otherwise stated, the rows of a $\mathrm{GBTP}_\lambda(K;v,m\times n)$ are indexed by $[m]$ and the columns by $[n]$.

In a $\mathrm{GBTP}_\lambda(K;v,m\times n)$, given point x and column j, there is a unique row that contains the point x in column j. Hence, for each point $x\in X$ of a $\mathrm{GBTP}_\lambda(K;v,m\times n)$ $(X,\mathcal{A})$, we may correspond the codeword $\mathsf{c}(x) = (r_1,r_2,\ldots,r_n)\in[m]^n$, where r_j is the row in which point x appears in column j. It is obvious that $\mathcal{C} = \{\mathsf{c}(x) : x\in X\}$ is an m-ary code of length n over the alphabet $[m]$. We note that this correspondence is precisely the one used by Semakov and Zinoviev [36] to show the equivalence between *equidistant codes* and resolvable balanced incomplete block designs.

For distinct points $x, y \in X$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is the number of columns for which x and y are not both contained in the same row. Since there are at most λ blocks containing both x and y, and that no two such blocks can occur in the same column of the $\mathrm{GBTP}_\lambda(K; v, m \times n)$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is at least $n - \lambda$.

Next, we determine $w_i(\mathsf{c}(x))$, for $x \in X$ and $i \in [m]$. From the construction of $\mathsf{c}(x)$, the number of times a symbol i appears in $\mathsf{c}(x)$ is the number of cells in row i that contains x. By the definition of a $\mathrm{GBTP}_\lambda(K; v, m \times n)$, this number belongs to $\{\lfloor n/m \rfloor, \lceil n/m \rceil\}$. Hence, $\mathcal{C}$ is an equitable symbol weight code with size v.

Finally, this construction of an equitable symbol weight code from a generalized balanced tournament packing can easily be reversed. We record these observations as:

Theorem 3.3. *Let K be set of nonnegative integers. Then there exists a* $\mathrm{GBTP}_\lambda(K; v, m \times n)$ *if and only if there exists an $(n, n-\lambda)_m$-equitable symbol weight code $\mathcal{C}$ of size v.*

Example 3.1. *Consider the $GBTP_1(\{2,3\}, 6, 3 \times 4)$ below.*

	$\{1,4\}$	$\{2,6\}$	$\{3,5\}$
$\{1,2,3\}$	$\{2,5\}$	$\{3,4\}$	$\{1,6\}$
$\{4,5,6\}$	$\{3,6\}$	$\{1,5\}$	$\{2,4\}$

Each point $x \in [6]$ gives a codeword $\mathsf{c}(x) = (r_1, r_2, \ldots, r_5)$, where r_j is the row in which point x appears in column j. Hence, we have

$$\begin{aligned}
\mathsf{c}(1) &= (2,1,3,2),\\
\mathsf{c}(2) &= (2,2,1,3),\\
\mathsf{c}(3) &= (2,3,2,1),\\
\mathsf{c}(4) &= (3,1,2,3),\\
\mathsf{c}(5) &= (3,2,3,1),\\
\mathsf{c}(6) &= (3,3,1,2).
\end{aligned}$$

The code $\mathcal{C} = \{\mathsf{c}(1), \mathsf{c}(2), \mathsf{c}(3), \mathsf{c}(4), \mathsf{c}(5), \mathsf{c}(6)\}$ is a $(4,3)_3$-equitable symbol weight code of size six.

Theorem 3.3 set up the equivalence between GBTPs and equitable symbol weight codes. In general, a GBTP may not correspond to an optimal equitable symbol weight code. However, in the following, we set K to specific values so as to derive families of optimal equitable symbol weight codes.

3.2. Optimal Equitable Symbol Weight Codes from Generalized Balanced Tournament Designs. A $\mathrm{GBTP}_\lambda\left(\{k\}; km, m \times \frac{\lambda(km-1)}{k-1}\right)$ is called a *generalized balanced tournament design* (GBTD), denoted by $\mathrm{GBTD}_\lambda(k, m)$. In this case, we check that each pair of distinct points is contained in exactly λ blocks and every point is contained in either $\left\lceil \frac{\lambda(km-1)}{m(k-1)} \right\rceil$ or $\left\lfloor \frac{\lambda(km-1)}{m(k-1)} \right\rfloor$ cells of each row.

Applying Theorem 3.3, a $\left(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1}\right)_m$-equitable symbol weight code of size km exists and the corresponding code is optimal by generalized Plotkin bound. We summarize in the following theorem.

Theorem 3.4. *There exists a $GBTD_\lambda(k, m)$ if and only if there exists an optimal $\left(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1}\right)_m$-equitable symbol weight code of size km which attains the generalized Plotkin bound.*

We remark that our definition of a generalized balanced tournament design extends that of Lamken [29], which corresponds in our definition to the case when $\lambda = k-1$.

The following theorem summarizes the state-of-the-art results on the existence of $\mathrm{GBTD}_{k-1}(k,m)$.

Theorem 3.5 (Lamken [29–31], Yin *et al.* [42], Chee *et al.* [11]). *The following holds.*

(i) A $GBTD_1(2,m)$ exists if and only if $m=1$ or $m \geq 3$.
(ii) A $GBTD_2(3,m)$ exists if and only if $m=1$ or $m \geq 3$.
(iii) A $GBTD_3(4,m)$ exists if and only if $m=1$ or $m \geq 4$.

Theorem 2.3(i), Theorem 2.3(iii) and Theorem 2.3(iv) is now an immediate consequence of Theorem 3.4, 3.5 and Proposition 3.1. The existence of $\mathrm{GBTD}_\lambda(k,m)$ when $\lambda \neq k-1$ has not been previously investigated. The smallest open case is when $k=3$ and $\lambda=1$, which is the case dealt with in this paper.

It follows readily from the fact that a $\mathrm{GBTD}_1(3,m)$ is also an $\mathrm{RBIBD}(3m,3,1)$, that a necessary condition for a $\mathrm{GBTD}_1(3,m)$ to exist is that m must be odd. We note from Proposition 3.1 that $A_3^{ESW}(4,3)=6$ and $A_5^{ESW}(7,6)=14$, which do not meet the Plotkin bound. Hence, the corresponding designs $\mathrm{GBTD}_1(3,3)$ and $\mathrm{GBTD}_1(3,5)$ do not exist by Theorem 3.4.

Hence, a $\mathrm{GBTD}_1(3,m)$ can exist only if m is odd and $m \notin \{3,5\}$. In Sections 5 to 8, we prove that this necessary condition is also sufficient for the existence of $\mathrm{GBTD}_1(3,m)$. A direct consequence of this is Theorem 2.3(v).

In general, when $k \geq 4$, a necessary condition for the existence of $\mathrm{GBTD}_\lambda(k,m)$ is that $\lambda(km-1) \equiv 0 \bmod k-1$. However, to show that this condition is also sufficient for all k and λ is difficult. Hence, for the general case, we construct infinite families of $\mathrm{GBTD}_\lambda(k,m)$ in Section 4. In particular, we show that for a fixed k and $\lambda \in [k-1]$, there exists an $\mathrm{GBTD}_\lambda(k,q)$ for sufficiently large prime power $q \equiv 1 \bmod k(k-1)$ and direct consequence is Theorem 2.3(vi).

3.3. Optimal Equitable Symbol Weight Codes from $\mathbf{GBTP}_1(\{2,3^*\};2m+1,m\times(2m-3))$. Theorem 3.4 constructs optimal equitable symbol weight codes from GBTDs. In this section, we make slight variations to obtain another infinite family of optimal equitable symbol weight codes.

Consider a $\mathrm{GBTP}_1(\{2,3\};v,m\times n)$. If there is exactly one block of size 3 in each resolution class, then we denote the GBTP by $\mathrm{GBTP}_1(\{2,3^*\};v,m\times n)$. A simple computation then shows $v=2m+1$. Now we establish the following construction for optimal equitable symbol weight codes.

Theorem 3.6. *Let $m \geq 7$. If there exists a $GBTP_1(\{2,3^*\};2m+1,m\times(2m-3))$, then there exists an optimal $(2m-3,2m-4)_m$-equitable symbol weight code of size $2m+1$ which attains the generalized Plotkin bound.*

Proof. By Theorem 3.3, we have a $(2m-3,2m-4)_m$-equitable symbol weight code of size $2m+1$. It remains to verify its optimality.

Suppose otherwise that there exists a $(2m-3,2m-4)_m$-equitable symbol weight code of size $2m+2$. Consider (2.1) in Theorem 2.2. On the left hand side, we have

$$\binom{2m+2}{2} \cdot (2m-4) = 4m^3 - 2m^2 - 10m - 4.$$

Since $\lfloor \frac{2m+2+i}{m} \rfloor = 2$ for $0 \leq i \leq m-3$ and $\left\lfloor \frac{2m+2+(m-2)}{m} \right\rfloor = \left\lfloor \frac{2m+2+(m-1)}{m} \right\rfloor = 3$, the

term on the right hand is

$$\begin{aligned}
&(2m-3)\left(\left(\sum_{i=0}^{m-3} 4(m-3-i)+12\right)+9\right)\\
&=(2m-3)(4m(m-2)-2(m-3)(m-2)+9)\\
&=4m^3-2m^2-12m+9
\end{aligned}$$

But for $m \geq 7$,

$$4m^3-2m^2-10m-4 > 4m^3-2m^2-12m+9,$$

contradicting (2.1). Hence, a $(2m-3, 2m-4)_m$-equitable symbol weight code of size $2m+2$ does not exist and the result follows. ☐

In Sections 5 to 8, we show the existence of a $\mathrm{GBTP}_1(\{2,3^*\}; 2m+1, m\times(2m-3))$ for $m \geq 4$, except possibly $m \in \{12, 13\}$. This with Theorem 3.6 and Proposition 3.1 gives Theorem 2.3(ii).

4. Infinite Families of Generalized Balanced Tournament Designs. In this section, we present a direct construction of GBTDs and as a result, exhibit the existence of infinite families of GBTDs. The main tool in our construction is the *method of differences*.

Let Γ be an additive abelian group and let n be a positive integer. For a set system $(\Gamma, \mathcal{S})$, the *difference list* of $\mathcal{S}$ is the multiset

$$\Delta\mathcal{S} = \langle x-y : x, y \in A,\ x \neq y, \text{ and } A \in \mathcal{S}\rangle.$$

For a set-system $(\Gamma \times [n], \mathcal{S})$, the multiset

$$\Delta_{ij}\mathcal{S} = \langle x-y : x_i, y_j \in A,\ x_i \neq y_j, \text{ and } A \in \mathcal{S}\rangle$$

is called a list of *pure differences* when $i = j$, and called a list of *mixed differences* when $i \neq j$ for $i, j \in [n]$.

Definition 4.1 (Starter for GBTD). *Let m be an odd positive integer, Γ be an additive abelian group of size m. Let T be an index set of size $(m-1)/(k-1)$. Let $(\Gamma \times [k], \mathcal{S})$ be a $\{k\}$-uniform set system of size $(km-1)/(k-1)$, where*

$$\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}.$$

$\mathcal{S}$ is called a $(\Gamma \times [k])$-GBTD-starter *if the following conditions hold:*

(i) $\Delta_{ii}\mathcal{S} = \Gamma \setminus \{0\}$, *for* $i \in [k]$,
(ii) $\Delta_{ij}\mathcal{S} = \Gamma$, *for* $i, j \in [k]$, $i \neq j$,
(iii) $\cup_{\alpha\in\Gamma} A_\alpha = \Gamma \times [k]$,
(iv) $\{j : \alpha_j \in B_t$ *for some* $\alpha \in \Gamma\} = [k]$, *for* $t \in T$,
(v) each element in $\Gamma \times [k]$ *appears either once or twice in the multiset*

$$R = \left(\bigcup_{\alpha\in\Gamma} A_\alpha - \alpha\right) \cup \left(\bigcup_{t\in T} B_t\right).$$

$$\boxed{\begin{array}{c|c} \mathsf{A} & \mathsf{B} \end{array}}$$

where A is the array

$$\boxed{\begin{array}{ccccc}
A_0 & A_{-\alpha_1}+\alpha_1 & A_{-\alpha_2}+\alpha_2 & \cdots & A_{-\alpha_{m-1}}+\alpha_{m-1} \\
A_{\alpha_1} & A_0+\alpha_1 & A_{\alpha_1-\alpha_2}+\alpha_2 & \cdots & A_{\alpha_1-\alpha_{m-1}}+\alpha_{m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{\alpha_{m-1}} & A_{\alpha_{m-1}-\alpha_1}+\alpha_1 & A_{\alpha_{m-1}-\alpha_2}+\alpha_2 & \cdots & A_0+\alpha_{m-1}
\end{array}}$$

and B is the array

$$\boxed{\begin{array}{cccc}
B_1 & B_2 & \cdots & B_{(m-1)/(k-1)} \\
B_1+\alpha_1 & B_2+\alpha_1 & \cdots & B_{(m-1)/(k-1)}+\alpha_1 \\
\vdots & \vdots & \ddots & \vdots \\
B_1+\alpha_{m-1} & B_2+\alpha_{m-1} & \cdots & B_{(m-1)/(k-1)}+\alpha_{m-1}
\end{array}}\,.$$

FIG. 4.1. *A* $\mathrm{GBTD}_1(k,m)$ *from* $(\Gamma\times[k])$*-GBTD-starter* $\mathcal{S}=\{A_\alpha:\alpha\in\Gamma\}\cup\{B_t:t\in T\}$, *where* $\Gamma=\{0,\alpha_1,\ldots,\alpha_{m-1}\}$ *and* $T=[(m-1)/(k-1)]$.

PROPOSITION 4.2. *If there exists a* $(\Gamma\times[k])$*-GBTD-starter, then there exists a* $\mathrm{GBTD}_1(k,m)$.

Proof. Let $X=\Gamma\times[k]$, and suppose $\mathcal{S}=\{A_\alpha:\alpha\in\Gamma\}\cup\{B_t:t\in T\}$ is an $(\Gamma\times[k])$-GBTD-starter. Let

$$\mathcal{A}=\bigcup_{A\in\mathcal{S}}\{A+\alpha:\alpha\in\Gamma\}.$$

Then $(X,\mathcal{A})$ is a $\mathrm{BIBD}(km,k,1)$, whose blocks can be arranged in an $m\times\frac{(km-1)}{k-1}$ array, whose rows and columns are indexed by Γ and $\Gamma\cup T$, respectively, as follows:

(i) for $\alpha,\beta\in\Gamma$, the block $A_\alpha+\beta$ is placed in cell $(\alpha+\beta,\beta)$, and

(ii) for $t\in T$ and $\alpha\in\Gamma$, the block $B_t+\alpha$ is placed in cell (α,t).

Fig. 4.1 depicts the placement of blocks in the array.

For $\beta\in\Gamma$, the set of blocks occupying column β is $\{A_\alpha+\beta:\alpha\in\Gamma\}$, which form a resolution class by condition (iii) of Definition 4.1. Similarly, for $t\in T$, the set of blocks occupying column t is $\{B_t+\alpha:\alpha\in\Gamma\}$, which form a resolution class by condition (iv) in Definition 4.1.

The set of blocks occupying row 0 is given by R, and by condition (v) of Definition 4.1, each point in X appears either once or twice in row 0. Since the blocks occupying row α ($\alpha\in\Gamma$) are exactly the translates of the blocks in R by α, every point in X also appears either once or twice in row α. ∎

DEFINITION 4.3 (Base Set for GBTD-starter). *Let q be prime power such that $q\equiv 1 \bmod k(k-1)$ with $s=(q-1)/(k(k-1))$. Let $\lambda\in[k-1]$. A* (q,k,λ)-base set *for a GBTD-starter is a set of $k+1+\lambda$ nonzero elements of $\mathbb{F}_q$, namely,*

$$\{a_i:i\in[k]\}\cup\{\theta\}\cup\{c_l:l\in[\lambda]\}$$

such that the following conditions hold:

(I) Define the following $\{k\}$-uniform set system $(\mathbb{F}_q, \mathcal{D})$ of size s, where

$$\mathcal{D} = \{\{\theta^t a_i : i \in [k]\} : t \in \mathbb{Z}_s\}.$$

Then,
(A) $\Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}$,
(B) the blocks in $\mathcal{D}$ are mutually disjoint.

(II) For $l \in [\lambda]$, $i \neq i' \in [k]$,

$$\frac{a_i + c_l}{a_{i'} + c_l} \notin \{0\} \cup \{\theta^t : t \in [s-1]\}.$$

(III) For $l \neq l' \in [\lambda]$, $i, i' \in [k]$,

$$\frac{a_i + c_l}{a_{i'} + c_{l'}} \notin \{0\} \cup \{\theta^t : t \in \mathbb{Z}_s\}.$$

Proposition 4.4. *Let q be prime power such that $q \equiv 1 \bmod k(k-1)$. If there exists a (q, k, λ)-base set for a GBTD-starter, then there exists an $(\mathbb{F}_q \times [k])$-GBTD-starter. Furthermore, there exists a $GBTD_\lambda(k, m)$.*

Proof. Assume the notations in Definition 4.3 and we first prove the existence of an $(\mathbb{F}_q \times [k])$-GBTD-starter. In fact, for each $l \in [\lambda]$, we construct an $(\mathbb{F}_q \times [k])$-GBTD-starter with the set system $\mathcal{D}$ and element c_l.

Define

$$\Lambda^{(l)} = \left\{-c_l \theta^t a_i : t \in \mathbb{Z}_s, i \in [k]\right\}$$

and construct the following $q + ks = (kq-1)/(k-1)$ blocks.

For $\alpha \in \mathbb{F}_q$,

$$A_\alpha^{(l)} = \begin{cases} \left\{\left(\theta^t a_i a_j, i\right) : j \in [k]\right\}, & \text{if } \alpha = -c_l \theta^t a_i \in \Lambda^{(l)} \text{ where } (t,i) \in \mathbb{Z}_s \times [k], \\ \left\{\left(-\frac{\alpha}{c_l} a_i, i\right) : i \in [k]\right)\right\}, & \text{otherwise.} \end{cases}$$

For $(t, j) \in [s] \times [k]$,

$$B_{(t,j)}^{(l)} = \left\{\left((a_i + c_l)\theta^t a_j, i\right) : i \in [3]\right\}.$$

Let $\mathcal{S}^{(l)} = \{A_\alpha^{(l)} : \alpha \in \mathbb{F}_q\} \cup \{B_{(t,j)}^{(l)} : (t,j) \in [s] \times [k]\}$ and we claim that $\mathcal{S}^{(l)}$ is a $(\mathbb{F}_q \times [k])$-GBTD-starter.

Indeed, for condition (i) of Definition 4.1, we check for $i \in [k]$,

$$\begin{aligned} \Delta_{ii}\mathcal{S}^{(l)} &= \Delta_{ii}\{A_\alpha^{(l)} : \alpha = -c_l\theta^t a_i, t \in \mathbb{Z}_s, i \in [k]\} \\ &= \theta^t a_i \Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}. \end{aligned}$$

For condition (ii),we verify for $i \neq i' \in [k]$,

$$\begin{aligned} &\Delta_{ii'}\mathcal{S}^{(l)} \\ &= \bigcup_{\alpha \notin \Lambda^{(l)}} \left(-\frac{\alpha}{c_l}(a_i - a_{i'})\right) \cup \bigcup_{(t,j) \in \mathbb{Z}_s \times [k]} \left(\theta^t a_j (a_i - a_{i'})\right) \\ &= (a_i - a_{i'}) \left(\bigcup_{\alpha \notin \Lambda^{(l)}} -\frac{\alpha}{c_l} \cup \bigcup_{(t,j) \in \mathbb{Z}_s \times [k]} \theta^t a_j\right) \\ &= (a_i - a_{i'})\, \mathbb{F}_q = \mathbb{F}_q. \end{aligned}$$

For condition (iii) of Definition 4.1, since the number of points in $\bigcup_{\alpha\in\mathbb{F}_q} A_\alpha^{(l)}$ is kq, it suffices to check that each point $(\beta,i)\in\mathbb{F}_q\times[k]$ belongs to some block $A_\alpha^{(l)}$.

Indeed, if $\beta/a_i=\theta^t a_j$ for some $(t,j)\in\mathbb{Z}_s\times[k]$, then let $\alpha=-c_l\theta^t a_i$ and so, $(\beta,i)=(\theta^t a_i a_j,i)$ belongs to $A_\alpha^{(l)}$. Otherwise, $-c_l\beta/a_i\notin\Lambda^{(l)}$. Let $\alpha=-c_l\beta/a_i$ and $(\beta,i)=(-(\alpha/c_l)a_i,i)\in A_\alpha^{(l)}$ as desired.

Condition (iv) of Definition 4.1 is clearly true from the definition of $B_{(t,j)}$. We establish condition (v) of Definition 4.1 through the following claims:

Claim 4.1. *The blocks in* $\bigcup_{\alpha\notin\Lambda^{(l)}}(A_\alpha^{(l)}-\alpha)\cup\bigcup_{(t,j)\in\mathbb{Z}_s\times[k]} B_{(t,j)}^{(l)}$ *form a resolution class.*

As above, it suffices to check that each point $(\beta,i)\in\mathbb{F}_q\times[k]$ belongs to some block in $\bigcup_{\alpha\notin\Lambda^{(l)}}(A_\alpha^{(l)}-\alpha)\cup\bigcup_{(t,j)\in\mathbb{Z}_s\times[k]} B_{(t,j)}^{(l)}$ as the total number of points is kq.

Indeed, if $\beta/(a_i+c_l)=\theta^t a_j$ for some $(t,j)\in\mathbb{Z}_s\times[k]$, then $(\beta,i)\in B_{(t,j)}^{(l)}$. Otherwise, $-\beta c_l/(a_i+c_l)\notin\Lambda^{(l)}$. Let $\alpha=-\beta c_l/(a_i+c_l)$ (note that α is well-defined by Condition (II) of Definition 4.3) and $(\beta,i)=(-\alpha\,(a_i/c_l+1)\,,i)\in A_\alpha^{(l)}-\alpha$.

Claim 4.2. *Each point in* $\mathbb{F}_q\times[k]$ *appears at most once in* $\bigcup_{\alpha\in\Lambda^{(l)}}\left(A_\alpha^{(l)}-\alpha\right)$.

Note that the blocks are of the form

$$\left\{\left((a_j+c_l)\theta^t a_i,i\right):j\in[k]\right\}$$

for $(t,i)\in\mathbb{Z}_s\times[k]$. Suppose otherwise that a point appears twice. That is, there exist $j,j'\in[k]$, $(t,i),(t',i)\in\mathbb{Z}_s\times[k]$ with $t>t'$ such that

$$(a_j+c_l)\theta^t a_i=(a_{j'}+c_l)\theta^{t'}a_i.$$

Hence,

$$(a_{j'}+c_l)/(a_j+c_l)=\theta^{t-t'}.$$

Since $t\neq t'$, hence, $j\neq j'$ and this contradicts Condition (II).

Therefore, $(\mathbb{F}_q\times[k],\mathcal{S}^{(l)})$ is a $(\mathbb{F}_q\times[k])$-GBTD-starter and a $\mathrm{GBTD}_1(k,m)$ $(\mathbb{F}_q\times[k],\mathcal{A}^{(l)})$ exists by Proposition 4.2 for each $l\in[\lambda]$.

Consider $\mathcal{A}^*=\bigcup_{l\in[\lambda]}\mathcal{A}^{(l)}$ and we claim that $(\mathbb{F}_q\times[k],\mathcal{A}^*)$ forms a $\mathrm{GBTD}_\lambda(k,q)$ by juxtapositioning the λ arrays $(\mathbb{F}_q\times[k],\mathcal{A}^{(l)})$ for $l\in[\lambda]$.

It follows immediately that $(\mathbb{F}_q\times[k],\mathcal{A}^*)$ is a $\mathrm{BIBD}(v,k,\lambda)$ and each point in $\mathbb{F}_q\times[k]$ appears exactly once in each column. Since $\lambda\in[k-1]$, $\left\lceil\frac{\lambda(kq-1)}{q(k-1)}\right\rceil=\lambda+1$ and $\left\lfloor\frac{\lambda(kq-1)}{q(k-1)}\right\rfloor=\lambda$, it remains to show that each point appears either λ or $\lambda+1$ times in $\mathbb{F}_q$.

To do so, we establish the following

Claim 4.3. *Each point in* $\mathbb{F}_q\times[k]$ *appears at most once in*

$$\bigcup_{l\in[\lambda]}\bigcup_{\alpha\in\Lambda^{(l)}}\left(A_\alpha^{(l)}-\alpha\right).$$

As in Claim 4.2, if a point appears twice, then there exists $l \neq l' \in [l]$, $j, j' \in [k]$, $(t, i), (t', i) \in \mathbb{Z}_s \times [k]$ with $t \geq t'$ such that

$$(a_j + c_l)\theta^t a_i = (a_{j'} + c_{l'})\theta^{t'} a_i$$

Hence,

$$(a_{j'} + c_{l'})/(a_j + c_l) = \theta^{t-t'},$$

contradicting Condition (III). ∎

Therefore, Proposition 4.2 and Proposition 4.4 establish that it suffices to construct a (q, k, λ)-base set to construct a $\mathrm{GBTD}_\lambda(k, q)$. In the rest of this section, we show that provided q is sufficiently large, a (q, k, λ)-base set can always be constructed.

The construction is based on the following two lemmas.

Lemma 4.5 (Chang [5]). *Let q be a prime power such that $q \equiv 1 \bmod k(k-1)$ with $k \geq 3$. Define $s = (q-1)/k(k-1)$ and let θ be an element of order s. Then for sufficiently large q, there exist nonzero elements $a_1, a_2, \ldots, a_k \in \mathbb{F}_q$, satisfying the following conditions:*
(i) $\Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}$, where $\mathcal{D} = \{\{\theta^t a_i : i \in [k]\} : t \in \mathbb{Z}_s\}$,
(ii) the blocks in $\mathcal{D}$ are mutually disjoint.

Suppose that $k \geq 3$ and $0 \leq i \leq k(k-1) - 1$. Define

$$G_i = \{\omega^t : t \equiv i \bmod k(k-1)\}$$

and let $\mathcal{G} = \{G_i : i \in \mathbb{Z}_{k(k-1)}\}$.

Lemma 4.6 (Chang and Ji [6], Buratti and Pasotti [4]). *Fix $r \geq 2$, any r-tuple $(a_1, a_2, \ldots, a_r) \in \mathbb{F}_q^r$ and r-tuple $(i_1, i_2, \ldots, i_r) \in \mathbb{Z}_{k(k-1)}^r$. Then there exists an $c \in \mathbb{F}_q^r$ such that*

$$a_j + c \in G_{i_j} \text{ for } j \in [r],$$

provided q is sufficiently large and $q \equiv 1 \bmod k(k-1)$.

Now we are in the position to present our main result in this section.

Proposition 4.7. *Let $k \geq 4$ and $\lambda \in [k-1]$. Then there is a constant $Q(k)$ such that a (q, k, λ)-base set exists for all prime power $q \geq Q(k)$ and $q \equiv 1 \bmod k(k-1)$.*

Proof. Let $s = (q-1)/k(k-1)$ and θ be an element of order s. By Lemma 4.5, there exists a constant $Q(k)$ such that when $q > Q(k)$, there exist nonzero elements $a_1, a_2, \ldots, a_k \in \mathbb{F}_q$ satisfying the following conditions:
(i) $\Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}$, where $\mathcal{D} = \{\{\theta^t a_i : i \in [k]\} : t \in \mathbb{Z}_s\}$,
(ii) the blocks in $\mathcal{D}$ are mutually disjoint.

Now we fix $l \in [\lambda]$ and consider the k-tuples $(a_1, a_2, \ldots, a_k)$ and $(lk, lk+1, \ldots, (l+1)k-1)$. By Lemma 4.6, there exists a $c_l \in \mathbb{F}_q$ such that $a_j + c_l \in G_{lk+j-1}$ for all $j \in [k]$. Then we observe the following.

First, $a_i + c_l \neq 0$ since $a_i + c_l \in G_{lk+i-1}$ for $i \in [k]$, $l \in [k-1]$.

Next, suppose that

$$\frac{a_i + c_l}{a_{i'} + c_{l'}} = \theta^t,$$

for some $t \in \mathbb{Z}_s$ and $(i,l) \neq (i',l') \in [k] \times [k-1]$. Then $a_i + c_l$ and $a_{i'} + c_{l'}$ belongs to some coset of G_0. This contradicts the fact that $a_i + c_l$ and $a_{i'} + c_{l'}$ belong to different cosets.

Hence, there exist $c_1, c_2, \ldots, c_{k-1} \in \mathbb{F}_q$ such that

$$\frac{a_i + c_l}{a_{i'} + c_l} \notin \{0\} \cup \{\theta^t : t \in [s-1]\}.$$

for any $i \neq i' \in [k]$ and

$$\frac{a_i + c_l}{a_{i'} + c_{l'}} \notin \{0\} \cup \{\theta^t : t \in \mathbb{Z}_s\}.$$

for any $l \neq l' \in [\lambda]$ and $i, i' \in [k]$.

Thus by Definition 4.3,

$$\{a_i : i \in [k]\} \cup \{\theta\} \cup \{c_l : l \in [\lambda]\}$$

form a (q,k,λ)-base set for a GBTD-starter. □

Combining Proposition 4.4 and Proposition 4.7, we obtain the following result of the existence of GBTDs of large order.

THEOREM 4.8. *Let $k \geq 4$ and $\lambda \in [k-1]$. For sufficiently large prime power $q \equiv 1 \bmod k(k-1)$, there exists a* $\mathrm{GBTD}_\lambda(k,q)$.

This completes the proof of Theorem 2.3(vi). We provide a stronger result for $k = 3$ in section 7.

5. Proof Strategy of Theorem 2.3(ii) and Theorem 2.3(v). In previous section, we demonstrated the existence of GBTDs for sufficiently large prime power order and hence, correspondingly, we obtain optimal equitable symbol weight codes. For the rest of the paper, we study GBTPs and GBTDs with certain special parameters. Specifically, we determine completely the existence of $\mathrm{GBTD}_1(3,m)$ and $\mathrm{GBTP}_1(\{2,3^*\}; 2m+1, m \times (2m-3))$.

Our proof is technical and rather complex. This section outlines the general strategy used, and introduces some required combinatorial designs.

As with most combinatorial designs, direct construction to settle their existence is often difficult. Instead, we develop a set of recursive constructions, building big designs from smaller ones. Direct methods are used to construct a large enough set of small designs on which the recursions can work to generate all larger designs. For our recursion techniques to work, the generalized balanced tournament packing must possess more structure than stipulated in its definition. First, we consider $\mathrm{GBTD}_1(3,m)$s that are $*$*colorable* which are defined below.

5.1. c-$*$colorable Generalized Balanced Tournament Designs.

DEFINITION 5.1. *Let c be a positive integer. A* c-$*$colorable $\mathrm{RBIBD}(v,k,\lambda)$ *is an* $\mathrm{RBIBD}(v,k,\lambda)$ *with the property that its $\frac{\lambda v(v-1)}{k(k-1)}$ blocks can be arranged in a $\frac{v}{k} \times \frac{\lambda(v-1)}{k-1}$ array, and each block can be colored with one of c colors so that*

(i) each point appears exactly once in each column, and

(ii) in each row, blocks of the same color are pairwise disjoint.

DEFINITION 5.2. *A $\mathrm{GBTD}_\lambda(k,m)$ is* c-$*$colorable *if each of its blocks can be colored with one of c colors so that in each row, blocks of the same color are pairwise disjoint.*

$0_0 0_1 \infty$ ♣	$2_0 4_0 3_1$ ♣	$6_1 4_0 1_0$ $\diamondsuit$	$2_1 1_1 6_1$ ♣	$5_1 1_0 2_0$ $\heartsuit$	$4_1 3_1 1_1$ $\heartsuit$	$5_1 4_1 2_1$ $\diamondsuit$
$6_1 5_1 3_1$ ♣	$1_0 1_1 \infty$ ♣	$3_0 5_0 4_1$ ♣	$3_0 3_1 \infty$ $\diamondsuit$	$5_0 0_0 6_1$ $\diamondsuit$	$6_0 1_0 0_1$ $\diamondsuit$	$0_0 2_0 1_1$ $\heartsuit$
$1_0 3_0 2_1$ ♣	$0_1 6_1 4_1$ ♣	$2_0 2_1 \infty$ $\diamondsuit$	$4_0 6_0 5_1$ ♣	$1_1 6_0 3_0$ $\diamondsuit$	$5_0 5_1 \infty$ $\heartsuit$	$3_1 1_0 5_0$ $\diamondsuit$
$4_1 2_0 6_0$ ♣	$5_1 3_0 0_0$ ♣	$1_1 0_1 5_1$ $\diamondsuit$	$0_1 5_0 2_0$ $\heartsuit$	$4_0 4_1 \infty$ $\diamondsuit$	$2_1 0_0 4_0$ $\heartsuit$	$6_0 6_1 \infty$ $\heartsuit$
$1_1 4_0 5_0$ ♣	$2_1 5_0 6_0$ $\diamondsuit$	$3_1 6_0 0_0$ ♣	$4_1 0_0 1_0$ $\diamondsuit$	$3_1 2_1 0_1$ $\heartsuit$	$6_1 2_0 3_0$ ♣	$0_1 3_0 4_0$ $\diamondsuit$

FIG. 5.1. *A 3-∗colorable RBIBD$(15,3,1)$ $(X,\mathcal{A})$, where $X = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$. The set of colors used is $\{♣, \diamondsuit, \heartsuit\}$. $(X,\mathcal{A})$ has property Π as 1_0 is a witness for ♣ and ∞ is a witnesses for both $\diamondsuit$ and $\heartsuit$ in row 1. For succinctness, a block $\{x,y,z\}$ is written xyz*

DEFINITION 5.3. *A k-∗colorable* RBIBD$(v,k,1)$ *is* k-∗colorable with property Π *if there exists a row r such that for each color i, there exists a point (called a* witness *for i) that is not contained in any block in row r that is colored i.*

A GBTD$_1(k,m)$ that is *c-∗colorable with property* Π is similarly defined.

EXAMPLE 5.1. *The RBIBD$(15,3,1)$ in Fig. 5.1 is* 3-∗*colorable with property* Π.

PROPOSITION 5.4. *If an* RBIBD$(v,k,1)$ *is* $(k-1)$-∗*colorable, then it is k-∗colorable with property* Π.

Proof. Consider a $(k-1)$-∗colorable RBIBD$(v,k,1)$ with colors $c_1, c_2, \cdots, c_{k-1}$. There must exists a point, say x, that appears only once in the first row. Recolor the block that contains this point with color c_k. This new coloring shows that the RBIBD$(v,k,1)$ is k-∗colorable with property Π, since in row one, for each of the colors $c_1, c_2, \cdots, c_{k-1}$, the point x does not appear in a block that is colored any one of $c_1, c_2, \cdots, c_{k-1}$, and for color c_k, we can always consider a point not in the block colored c_k. □

EXAMPLE 5.2. *The* GBTD$_1(3,9)$ *in Fig. 5.2 is* 2-∗*colorable and is therefore* 3-∗*colorable with property* Π *by Proposition 5.4.*

5.2. Ingredient Generalized Balanced Tournament Packings. Suppose that $(X,\mathcal{A})$ is a (v,K,λ)-packing. Let $W \subset X$ with $|W| = w$. Furthermore, we call $(X,W,\mathcal{A})$ is an *ingredient resolvable packing*, denoted by IRP$(v,K,\lambda;w)$, if it satisfies the following conditions:

(i) any pair of points from W occurs in no blocks of $\mathcal{A}$,
(ii) the blocks in $\mathcal{A}$ can be partitioned into parallel classes and *partial parallel classes* $X \setminus W$.

DEFINITION 5.5. *Let $(X,W,\mathcal{A})$ be an IRP$(v,K,\lambda;w)$. Then $(X,W,\mathcal{A})$ is called an* ingredient generalized balanced tournament packing *(IGBTP) if the blocks of $\mathcal{A}$ are arranged into an $m \times n$ array* A*, with rows and columns indexed by R and C respectively, satisfying the following conditions:*

(i) there exist a $P \subset R$ with $|P| = m'$ and a $Q \subset C$ with $|Q| = n'$ such that the cell (r,c) is empty if $r \in P$ and $c \in Q$;
(ii) for any row $r \in P$, every point in $X \setminus W$ is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells and the points in W do not appear; for any row $r \in R \setminus P$, every point in X is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells;
(iii) the blocks in any column $c \in Q$ form a partial parallel class of $X \setminus W$ and the blocks in any column $c \in C \setminus Q$ forms a parallel class of X.

Denote such an IGBTP by IGBTP$_\lambda(K,v,m \times n;w,m' \times n')$.

EXAMPLE 5.3. *An* IGBTP$_1(\{2,3^*\},29,14 \times 25;9,4 \times 5)$ *is given in Fig. 5.3.*

A	B

where A is the array

$1_07_1\infty_2$ $\clubsuit$	$6_03_2\infty_1$ $\clubsuit$	$0_04_16_2$ $\clubsuit$	$6_07_10_2$ $\diamondsuit$	$7_00_11_2$ $\clubsuit$	$5_05_15_2$ $\diamondsuit$	$1_14_2\infty_0$ $\clubsuit$
$5_02_2\infty_1$ $\diamondsuit$	$2_00_1\infty_2$ $\clubsuit$	$7_04_2\infty_1$ $\clubsuit$	$1_05_17_2$ $\clubsuit$	$4_04_14_2$ $\diamondsuit$	$0_01_12_2$ $\clubsuit$	$7_05_1\infty_2$ $\diamondsuit$
$3_16_2\infty_0$ $\clubsuit$	$4_17_2\infty_0$ $\diamondsuit$	$3_01_1\infty_2$ $\clubsuit$	$0_05_2\infty_1$ $\clubsuit$	$2_06_10_2$ $\clubsuit$	$3_07_11_2$ $\diamondsuit$	$1_02_13_2$ $\clubsuit$
$3_04_15_2$ $\clubsuit$	$1_01_11_2$ $\diamondsuit$	$5_10_2\infty_0$ $\clubsuit$	$4_02_1\infty_2$ $\diamondsuit$	$5_03_1\infty_2$ $\clubsuit$	$2_07_2\infty_1$ $\clubsuit$	$4_00_12_2$ $\clubsuit$
$6_02_14_2$ $\diamondsuit$	$4_05_16_2$ $\clubsuit$	$5_06_17_2$ $\diamondsuit$	$6_11_2\infty_0$ $\clubsuit$	$1_06_2\infty_1$ $\diamondsuit$	$6_04_1\infty_2$ $\clubsuit$	$3_00_2\infty_1$ $\clubsuit$
$0_00_10_2$ $\diamondsuit$	$5_02_10_2$ $\clubsuit$	$4_07_13_2$ $\clubsuit$	$2_01_16_2$ $\clubsuit$	$7_12_2\infty_0$ $\diamondsuit$	$1_06_14_2$ $\clubsuit$	$0_03_17_2$ $\clubsuit$
$7_06_13_2$ $\clubsuit$	$7_03_15_2$ $\diamondsuit$	$6_03_11_2$ $\clubsuit$	$5_00_14_2$ $\clubsuit$	$3_02_17_2$ $\clubsuit$	$0_13_2\infty_0$ $\diamondsuit$	$2_07_15_2$ $\clubsuit$
$2_05_11_2$ $\clubsuit$	$0_07_14_2$ $\clubsuit$	$2_02_12_2$ $\diamondsuit$	$7_04_12_2$ $\clubsuit$	$6_01_15_2$ $\clubsuit$	$4_03_10_2$ $\clubsuit$	$6_06_16_2$ $\diamondsuit$
$4_01_17_2$ $\clubsuit$	$3_06_12_2$ $\clubsuit$	$1_00_15_2$ $\clubsuit$	$3_03_13_2$ $\diamondsuit$	$0_05_13_2$ $\clubsuit$	$7_02_16_2$ $\clubsuit$	$5_04_11_2$ $\clubsuit$

where B is the array

$0_06_1\infty_2$ $\diamondsuit$	$4_06_15_2$ $\clubsuit$	$1_02_04_0$ $\diamondsuit$	$2_03_05_0$ $\clubsuit$	$2_27_20_2$ $\clubsuit$	$2_13_15_1$ $\clubsuit$
$2_15_2\infty_0$ $\clubsuit$	$5_07_16_2$ $\clubsuit$	$3_14_16_1$ $\clubsuit$	$0_11_13_1$ $\diamondsuit$	$3_04_06_0$ $\clubsuit$	$3_20_21_2$ $\clubsuit$
$2_03_14_2$ $\diamondsuit$	$6_00_17_2$ $\clubsuit$	$4_21_22_2$ $\clubsuit$	$4_15_17_1$ $\clubsuit$	$5_16_10_1$ $\diamondsuit$	$4_05_07_0$ $\clubsuit$
$7_07_17_2$ $\diamondsuit$	$0_02_11_2$ $\clubsuit$	$0_25_26_2$ $\diamondsuit$	$6_07_01_0$ $\clubsuit$	$6_23_24_2$ $\clubsuit$	$6_17_11_1$ $\clubsuit$
$5_01_13_2$ $\clubsuit$	$1_03_12_2$ $\clubsuit$	$7_10_12_1$ $\clubsuit$	$5_22_23_2$ $\diamondsuit$	$7_00_02_0$ $\clubsuit$	$7_24_25_2$ $\clubsuit$
$6_05_12_2$ $\clubsuit$	$2_04_13_2$ $\diamondsuit$	$3_07_0\infty_0$ $\clubsuit$	$1_26_27_2$ $\diamondsuit$	$5_21_2\infty_2$ $\clubsuit$	$4_10_1\infty_1$ $\clubsuit$
$1_04_10_2$ $\clubsuit$	$3_05_14_2$ $\diamondsuit$	$1_15_1\infty_1$ $\clubsuit$	$4_00_0\infty_0$ $\clubsuit$	$1_12_14_1$ $\diamondsuit$	$6_22_2\infty_2$ $\clubsuit$
$3_00_16_2$ $\clubsuit$	$\infty_0\infty_1\infty_2$ $\diamondsuit$	$3_27_2\infty_2$ $\clubsuit$	$2_16_1\infty_1$ $\clubsuit$	$5_01_0\infty_0$ $\clubsuit$	$0_01_03_0$ $\diamondsuit$
$4_01_2\infty_1$ $\diamondsuit$	$7_01_10_2$ $\diamondsuit$	$5_06_00_0$ $\diamondsuit$	$4_20_2\infty_2$ $\clubsuit$	$3_17_1\infty_1$ $\clubsuit$	$6_02_0\infty_0$ $\clubsuit$

FIG. 5.2. *A 2-$*$colorable special* $\mathrm{GBTD}_1(3,9)$ $(X,\mathcal{A})$*, where* $X = (\mathbb{Z}_8 \times \mathbb{Z}_3) \cup \{\infty_0, \infty_1, \infty_2\}$ *and colors* $\{\clubsuit, \diamondsuit\}$*. The cell* $(1,5)$*, occupied by the block* $7_00_11_2$*, is special. For succinctness, a set* $\{x,y,z\}$ *is written* xyz*.*

Consider an $\mathrm{IGBTP}_1(\{k\}, km, m \times \frac{km-1}{k-1}; k, 1 \times 1)$. Then its corresponding array has one empty cell and we fill this cell with the block W to obtain a $\mathrm{GBTD}_1(k,m)$. A $\mathrm{GBTD}_1(k,m)$ obtained in this way is called a *special* $\mathrm{GBTD}_1(k,m)$ and the cell occupied by W is said to be *special*.

EXAMPLE 5.4. *The* $GBTD_1(3,9)$ *in Fig. 5.2 is a special* $GBTD_1(3,9)$ *with special cell* $(1,5)$.

A few more classes of auxiliary designs are also required.

5.3. Group Divisible Designs and Transversal Designs. DEFINITION 5.6. *Let* $(X,\mathcal{A})$ *be a set system and let* $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ *be a partition of* X *into subsets, called* groups*. The triple* $(X,\mathcal{G},\mathcal{A})$ *is a* group divisible design *(GDD) when every* 2*-subset of* X *not contained in a group appears in exactly one block, and* $|A \cap G| \leq 1$ *for all* $A \in \mathcal{A}$ *and* $G \in \mathcal{G}$.

We denote a GDD $(X,\mathcal{G},\mathcal{A})$ by K-GDD if $(X,\mathcal{A})$ is K-uniform. The *type* of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\langle |G| : G \in \mathcal{G} \rangle$. When more convenient, the exponential notation is used to describe the type of a GDD: a GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}$ is a GDD

A	B

where A is the array

–	–	–	–	–	$2,13$	$3,14$	$4,15$	$5,16$	$6,17$	$7,18$	$8,19$	$9,0$
–	–	–	–	–	$12,16$	$13,17$	$14,18$	$15,19$	$16,0$	$17,1$	$18,2$	$19,3$
–	–	–	–	–	$15,18$	$16,19$	$17,0$	$18,1$	$19,2$	$0,3$	$1,4$	$2,5$
–	–	–	–	–	$1,3$	$2,4$	$3,5$	$4,6$	$5,7$	$6,8$	$7,9$	$8,10$
$0,10$	$2,7$	$12,17$	$4,16$	$14,6$	$4,5,11$	$i,18$	$h,1$	$g,12$	$f,18$	$e,13$	$d,16$	$c,13$
$1,11$	$3,8$	$13,18$	$5,17$	$15,7$	$a,0$	$5,6,12$	$i,19$	$h,2$	$g,13$	$f,19$	$e,14$	$d,17$
$2,12$	$4,9$	$14,19$	$6,18$	$16,8$	$b,7$	$a,1$	$6,7,13$	$i,0$	$h,3$	$g,14$	$f,0$	$e,15$
$3,13$	$5,10$	$15,0$	$7,19$	$17,9$	$c,6$	$b,8$	$a,2$	$7,8,14$	$i,1$	$h,4$	$g,15$	$f,1$
$4,14$	$6,11$	$16,1$	$8,0$	$18,10$	$d,10$	$c,7$	$b,9$	$a,3$	$8,9,15$	$i,2$	$h,5$	$g,16$
$5,15$	$7,12$	$17,2$	$9,1$	$19,11$	$e,8$	$d,11$	$c,8$	$b,10$	$a,4$	$9,10,16$	$i,3$	$h,6$
$6,16$	$8,13$	$18,3$	$10,2$	$0,12$	$f,14$	$e,9$	$d,12$	$c,9$	$b,11$	$a,5$	$10,11,17$	$i,4$
$7,17$	$9,14$	$19,4$	$11,3$	$1,13$	$g,9$	$f,15$	$e,10$	$d,13$	$c,10$	$b,12$	$a,6$	$11,12,18$
$8,18$	$10,15$	$0,5$	$12,4$	$2,14$	$h,19$	$g,10$	$f,16$	$e,11$	$d,14$	$c,11$	$b,13$	$a,7$
$9,19$	$11,16$	$1,6$	$13,5$	$3,15$	$i,17$	$h,0$	$g,11$	$f,17$	$e,12$	$d,15$	$c,12$	$b,14$

where B is the array

$10,1$	$11,2$	$12,3$	$13,4$	$14,5$	$15,6$	$16,7$	$17,8$	$18,9$	$19,10$	$0,11$	$1,12$
$0,4$	$1,5$	$2,6$	$3,7$	$4,8$	$5,9$	$6,10$	$7,11$	$8,12$	$9,13$	$10,14$	$11,15$
$3,6$	$4,7$	$5,8$	$6,9$	$7,10$	$8,11$	$9,12$	$10,13$	$11,14$	$12,15$	$13,16$	$14,17$
$9,11$	$10,12$	$11,13$	$12,14$	$13,15$	$14,16$	$15,17$	$16,18$	$17,19$	$18,0$	$19,1$	$0,2$
$b,15$	$a,9$	$14,15,1$	$i,8$	$h,11$	$g,2$	$f,8$	$e,3$	$d,6$	$c,3$	$b,5$	$a,19$
$c,14$	$b,16$	$a,10$	$15,16,2$	$i,9$	$h,12$	$g,3$	$f,9$	$e,4$	$d,7$	$c,4$	$b,6$
$d,18$	$c,15$	$b,17$	$a,11$	$16,17,3$	$i,10$	$h,13$	$g,4$	$f,10$	$e,5$	$d,8$	$c,5$
$e,16$	$d,19$	$c,16$	$b,18$	$a,12$	$17,18,4$	$i,11$	$h,14$	$g,5$	$f,11$	$e,6$	$d,9$
$f,2$	$e,17$	$d,0$	$c,17$	$b,19$	$a,13$	$18,19,5$	$i,12$	$h,15$	$g,6$	$f,12$	$e,7$
$g,17$	$f,3$	$e,18$	$d,1$	$c,18$	$b,0$	$a,14$	$19,0,6$	$i,13$	$h,16$	$g,7$	$f,13$
$h,7$	$g,18$	$f,4$	$e,19$	$d,2$	$c,19$	$b,1$	$a,15$	$0,1,7$	$i,14$	$h,17$	$g,8$
$i,5$	$h,8$	$g,19$	$f,5$	$e,0$	$d,3$	$c,0$	$b,2$	$a,16$	$1,2,8$	$i,15$	$h,18$
$12,13,19$	$i,6$	$h,9$	$g,0$	$f,6$	$e,1$	$d,4$	$c,1$	$b,3$	$a,17$	$2,3,9$	$i,16$
$a,8$	$13,14,0$	$i,7$	$h,10$	$g,1$	$f,7$	$e,2$	$d,5$	$c,2$	$b,4$	$a,18$	$3,4,10$

FIG. 5.3. *An* $\mathrm{IGBTP}_1(\{2,3\},29,14\times 25;9,4\times 5)$ $(X,\mathcal{A})$*, where* $X=\mathbb{Z}_{20}\cup\{a,b,c,d,e,f,g,h,i\}$ *and* $W=\{a,b,c,d,e,f,g,h,i\}$*. For succinctness, a block* $\{x,y,z\}$ *is written* x,y,z*.*

where there are exactly t_i groups of size g_i, $i \in [s]$.

DEFINITION 5.7. *A* transversal design *TD*(k,n) *is a* $\{k\}$*-GDD of type* n^k*.*

The following result on the existence of transversal designs (see [1]) is sometimes used without explicit reference throughout this paper.

THEOREM 5.8. *Let* $\mathrm{TD}(k)$ *denote the set of positive integers* n *such that there exists a* $\mathrm{TD}(k,n)$*. Then, we have*

A	B

where A is the array

–	–	–	$4_0 1_0 7_0$	$4_1 1_1 7_1$	$4_2 1_2 7_2$	$6_0 3_0 9_0$	$6_1 3_1 9_1$	$6_2 3_2 9_2$
–	–	–	$6_0 7_2 8_0$	$6_1 7_0 8_1$	$6_2 7_1 8_2$	$8_0 9_2 0_0$	$8_1 9_0 0_1$	$8_2 9_1 0_2$
$2_0 8_1 1_0$	$2_1 8_2 1_1$	$2_2 8_0 1_2$	–	–	–	$4_1 8_2 \infty_4$	$4_2 8_0 \infty_3$	$4_0 8_1 \infty_5$
$6_2 7_2 3_1$	$6_0 7_0 3_2$	$6_1 7_1 3_0$	–	–	–	$9_1 1_2 \infty_5$	$9_2 1_0 \infty_4$	$9_0 1_1 \infty_3$
$4_0 0_1 3_0$	$4_1 0_2 3_1$	$4_2 0_0 3_2$	$1_1 3_0 9_1$	$1_2 3_1 9_2$	$1_0 3_2 9_0$	–	–	–
$8_2 9_2 5_1$	$8_0 9_0 5_2$	$8_1 9_1 5_0$	$4_2 6_1 8_2$	$4_0 6_2 8_0$	$4_1 6_0 8_1$	–	–	–
$6_0 2_1 5_0$	$6_1 2_2 5_1$	$6_2 2_0 5_2$	$8_1 1_2 \infty_0$	$8_2 1_0 \infty_1$	$8_0 1_1 \infty_2$	$3_1 5_0 1_1$	$3_2 5_1 1_2$	$3_0 5_2 1_0$
$0_2 1_2 7_1$	$0_0 1_0 7_2$	$0_1 1_1 7_0$	$2_0 3_1 \infty_1$	$2_1 3_2 \infty_2$	$2_2 3_0 \infty_0$	$6_2 8_1 0_2$	$6_0 8_2 0_0$	$6_1 8_0 0_1$
$8_0 4_1 7_0$	$8_1 4_2 7_1$	$8_2 4_0 7_2$	$2_2 9_0 \infty_2$	$2_0 9_1 \infty_0$	$2_1 9_2 \infty_1$	$0_1 3_2 \infty_0$	$0_2 3_0 \infty_1$	$0_0 3_1 \infty_2$
$2_2 3_2 9_1$	$2_0 3_0 9_2$	$2_1 3_1 9_0$	$3_2 4_1 \infty_3$	$3_0 4_2 \infty_5$	$3_1 4_0 \infty_4$	$4_0 5_1 \infty_1$	$4_1 5_2 \infty_2$	$4_2 5_0 \infty_0$
$0_0 6_1 9_0$	$0_1 6_2 9_1$	$0_2 6_0 9_2$	$2_1 6_2 \infty_4$	$2_2 6_0 \infty_3$	$2_0 6_1 \infty_5$	$4_2 1_0 \infty_2$	$4_0 1_1 \infty_0$	$4_1 1_2 \infty_1$
$4_2 5_2 1_1$	$4_0 5_0 1_2$	$4_1 5_1 1_0$	$7_1 9_2 \infty_5$	$7_2 9_0 \infty_4$	$7_0 9_1 \infty_3$	$5_2 6_1 \infty_3$	$5_0 6_2 \infty_5$	$5_1 6_0 \infty_4$

where B is the array

$8_0 5_0 1_0$	$8_1 5_1 1_1$	$8_2 5_2 1_2$	$0_0 7_0 3_0$	$0_1 7_1 3_1$	$0_2 7_2 3_2$	$2_0 9_0 5_0$	$2_1 9_1 5_1$	$2_2 9_2 5_2$
$0_0 1_2 2_0$	$0_1 1_0 2_1$	$0_2 1_1 2_2$	$2_0 3_2 4_0$	$2_1 3_0 4_1$	$2_2 3_1 4_2$	$4_0 5_2 6_0$	$4_1 5_0 6_1$	$4_2 5_1 6_2$
$6_2 3_0 \infty_2$	$6_0 3_1 \infty_0$	$6_1 3_2 \infty_1$	$4_1 7_2 \infty_0$	$4_2 7_0 \infty_1$	$4_0 7_1 \infty_2$	$9_1 1_0 7_1$	$9_2 1_1 7_2$	$9_0 1_2 7_0$
$7_2 8_1 \infty_3$	$7_0 8_2 \infty_5$	$7_1 8_0 \infty_4$	$8_0 9_1 \infty_1$	$8_1 9_2 \infty_2$	$8_2 9_0 \infty_0$	$2_2 4_1 6_2$	$2_0 4_2 6_0$	$2_1 4_0 6_1$
$6_1 0_2 \infty_4$	$6_2 0_0 \infty_3$	$6_0 0_1 \infty_5$	$8_2 5_0 \infty_2$	$8_0 5_1 \infty_0$	$8_1 5_2 \infty_1$	$6_1 9_2 \infty_0$	$6_2 9_0 \infty_1$	$6_0 9_1 \infty_2$
$1_1 3_2 \infty_5$	$1_2 3_0 \infty_4$	$1_0 3_1 \infty_3$	$9_2 0_1 \infty_3$	$9_0 0_2 \infty_5$	$9_1 0_0 \infty_4$	$0_0 1_1 \infty_1$	$0_1 1_2 \infty_2$	$0_2 1_0 \infty_0$
–	–	–	$8_1 2_2 \infty_4$	$8_2 2_0 \infty_3$	$8_0 2_1 \infty_5$	$0_2 7_0 \infty_2$	$0_0 7_1 \infty_0$	$0_1 7_2 \infty_1$
–	–	–	$3_1 5_2 \infty_5$	$3_2 5_0 \infty_4$	$3_0 5_1 \infty_3$	$1_2 2_1 \infty_3$	$1_0 2_2 \infty_5$	$1_1 2_0 \infty_4$
$5_1 7_0 3_1$	$5_2 7_1 3_2$	$5_0 7_2 3_0$	–	–	–	$0_1 4_2 \infty_4$	$0_2 4_0 \infty_3$	$0_0 4_1 \infty_5$
$8_2 0_1 2_2$	$8_0 0_2 2_0$	$8_1 0_0 2_1$	–	–	–	$5_1 7_2 \infty_5$	$5_2 7_0 \infty_4$	$5_0 7_1 \infty_3$
$2_1 5_2 \infty_0$	$2_2 5_0 \infty_1$	$2_0 5_1 \infty_2$	$7_1 9_0 5_1$	$7_2 9_1 5_2$	$7_0 9_2 5_0$	–	–	–
$6_0 7_1 \infty_1$	$6_1 7_2 \infty_2$	$6_2 7_0 \infty_0$	$0_2 2_1 4_2$	$0_0 2_2 4_0$	$0_1 2_0 4_1$	–	–	–

FIG. 5.4. *An* $\mathrm{FGBTD}_1(3, 6^6)$ $(X, \mathcal{G}, \mathcal{A})$*, where* $X = (\mathbb{Z}_{10} \times \mathbb{Z}_3) \cup \{\infty_i : i \in \mathbb{Z}_6\}$ *and* $\mathcal{G} = \{\{t_0, t_1, t_2, (5+t)_0, (5+t)_1, (5+t)_2\} : t \in \mathbb{Z}_5\} \cup \{\infty_i : i \in \mathbb{Z}_6\}$*. For succinctness, a set* $\{x, y, z\}$ *is written* xyz*.*

(i) $\mathrm{TD}(4) \supseteq \mathbb{Z}_{>0} \setminus \{2, 6\}$,
(ii) $\mathrm{TD}(5) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 6, 10\}$,
(iii) $\mathrm{TD}(6) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 6, 10, 14, 18, 22\}$,
(iv) $\mathrm{TD}(7) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$,
(v) $\mathrm{TD}(k) \supseteq \{q : q \geq k - 1 \textit{ is a prime power}\}$.

DEFINITION 5.9. *A* doubly resolvable *TD*(k, n)*, denoted by DRTD*(k, n)*, is a TD*(k, n) *whose blocks can be arranged in an* $n \times n$ *array such that each point appears exactly once in each row and once in each column.*

Colbourn *et al.* [15] established the following.

Proposition 5.10 (Colbourn *et al.* [15]). *A TD$(k+2,n)$ exists if and only if a DRTD(k,n) exists.*

Corollary 5.11. *A DRTD$(3,n)$ exists for all $n \geq 4$ and $n \notin \{6,10\}$.*

Proof. A TD$(5,n)$ exists if $n \geq 4$ and $n \notin \{6,10\}$ by Theorem 5.8. ∎

5.4. Frame Generalized Balanced Tournament Design. Let $(X,\mathcal{G},\mathcal{A})$ be a $\{k\}$-GDD with $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ and $|G_i| \equiv 0 \bmod k(k-1)$ for all $i \in [s]$. Let $R = \frac{1}{k}\sum_{i=1}^{s} |G_i|$ and $C = \frac{1}{k-1}\sum_{i=1}^{s} |G_i|$. Suppose there exists a partition $[R] = \bigsqcup_{i=1}^{s} R_i$ and a partition $[C] = \bigsqcup_{i=1}^{s} C_i$ such that for each $i \in [s]$, we have $|R_i| = |G_i|/k$ and $|C_i| = |G_i|/(k-1)$.

We say that $(X,\mathcal{G},\mathcal{A})$ is a *frame generalized balanced tournament design* (FGBTD) if its blocks can be arranged in an $R \times C$ array such that the following conditions hold:

(i) the cell (r,c) is empty when $(r,c) \in R_i \times C_i$ for $i \in [s]$,

(ii) for any row $r \in R_i$, each point in $X \setminus G_i$ appears either once or twice and the points in G_i do not appear,

(iii) for any column $c \in C_i$, each point in $X \setminus G_i$ appears exactly once.

Denote this FGBTD by FGBTD(k,T), where $T = \langle |G_i| : i \in [s] \rangle$.

Example 5.5. *An* FGBTD$(3,6^6)$ *is given in Fig. 5.4.*

6. Recursive Constructions. In this section, we develop the necessary recursive constructions used.

6.1. Recursive Constructions for GBTPs. First, for block size three, we have the following tripling construction for GBTDs.

Proposition 6.1 (Tripling Construction). *Suppose there exist a* 3-$*$*colorable RBIBD$(m,3,1)$ and a DRTD$(3,m)$. Then there exists a* 2-$*$*colorable GBTD$_1(3,m)$. Suppose further that the RBIBD$(m,3,1)$ is* 3-$*$*colorable with property* Π. *Then the GBTD$_1(3,m)$ is a special GBTD$_1(3,m)$.*

Proof. Consider a 3-$*$colorable RBIBD$(m,3,1)$ $(X,\mathcal{A})$ with colors from $\mathbb{Z}_3$ and let

$$X' = \{x_i : x \in X \text{ and } i \in \mathbb{Z}_3\}.$$

Make three copies of the 3-$*$colorable RBIBD$(m,3,1)$ as follows: for the jth copy, $j \in \{1,2,3\}$, each block $\{x,y,z\}$ of color i in the 3-$*$colorable RBIBD$(m,3,1)$ is replaced by the block $\{x_{i+j}, y_{i+j}, z_{i+j}\}$, where arithmetic in the subscripts is performed modulo three. Stacking these three $\frac{m}{3} \times \frac{m-1}{2}$ arrays together gives an $m \times \frac{m-1}{2}$ array A with the property that

(i) each point in X' appears exactly once in each column,

(ii) each point in X' appears at most once in each row.

Now take a DRTD$(3,m)$ $(X',\mathcal{G},\mathcal{A})$, where

$$\mathcal{G} = \{\{x_i : x \in X\} : i \in \mathbb{Z}_3\},$$

and adjoin it to A. This gives an $m \times \frac{3m-1}{2}$ array, which we claim is a GBTD$_1(3,m)$. Indeed it is easy to see that in this array, each point in X' appears exactly once in each column and either once or twice in each row. It remains to show that this array is a BIBD$(3m,3,1)$. To see this, observe that any pair of points contained in a group of the DRTD$(3,m)$ is contained in a block of one of the copies of the 3-$*$colorable RBIBD$(m,3,1)$. This GBTD$_1(3,m)$ is 2-$*$colorable by giving the blocks

from the DRTD$(3, m)$ one color and the remaining blocks (from the three copies of the RBIBD$(m, 3, 1)$) another color.

If, in addition, the RBIBD$(m, 3, 1)$ is 3-$*$colorable with property Π, and that in row r of this RBIBD$(m, 3, 1)$, the points x, y, z (not necessarily distinct) are witnesses for colors $0, 1, 2$, respectively, then we can always assume that the DRTD$(3, m)$ used has the block $\{x_0, y_1, z_2\}$ and that this block can be made to appear in row r, by permuting rows if necessary. The cell that contains $\{x_0, y_1, z_2\}$ is a special cell of the GBTD$_1(3, m)$. $\square$

Corollary 6.2. *Let $m > 3$ and suppose there exists an RBIBD$(m, 3, 1)$ that is 3-$*$colorable with property Π. Then there exists a special GBTD$_1(3, 3^k m)$, for all $k \geq 0$.*

Proof. First note that $m \equiv 3 \bmod 6$ since this is a necessary condition for the existence of an RBIBD$(m, 3, 1)$. Hence, there exists a DRTD$(3, m)$ by Corollary 5.11. By Proposition 6.1, there exists a 2-$*$colorable special GBTD$_1(3, m)$, which may be regarded as an RBIBD$(3m, 3, 1)$ that is 3-$*$colorable with property Π. The corollary then follows by induction. $\square$

The following is a simple, but useful construction.

Proposition 6.3. *If there exists an IGBTP$_\lambda(K, v, m \times n; w, m' \times n')$ and a GBTP$_\lambda(K, w, m' \times n')$, then a GBTP$_\lambda(K, v, m \times n)$ exists.*

Proof. Let $(X, \mathcal{A})$ be an IGBTP$_\lambda(K, v, m \times n; w, m' \times n')$. Fill in the empty subarray of this IGBTP with an a GBTP$_\lambda(K, w, m' \times n')$, $(X', \mathcal{A}')$. The resulting array is a GBTP$_\lambda(K, v, m \times n)$, $(X, \mathcal{A} \cup \mathcal{A}')$. $\square$

FGBTD is a useful tool to construct larger GBTPs from smaller ones.

Proposition 6.4 (FGBTD Construction for GBTP). *Let $k \in K$. Suppose there exists an* FGBTD(k, T) $(X, \mathcal{G}, \mathcal{A})$*, where $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$, and let $r_i = |G_i|/k$ and $c_i = |G_i|/(k-1)$, for $i \in [s]$. If there exists an IGBTP$_1(K, |G_i|+w, (r_i+m)\times(c_i+n); w, m \times n)$ for all $i \in [s]$, then there exists an IGBTP$_1(K, \sum_{i=1}^s |G_i|+w, (\sum_{i=1}^s r_i + m) \times (\sum_{i=1}^s c_i + n); w, m \times n)$. Furthermore, if a GBTP$_1(K, w, m \times n)$ exists, then an GBTP$_1(K, \sum_{i=1}^s |G_i| + w, (\sum_{i=1}^s r_i + m) \times (\sum_{i=1}^s c_i + n))$ exists.*

Proof. We use the notations as in the definition of FGBTD in Section 5.4, and assume that the blocks of the FGBTD(k, T) are arranged in an $R \times C$ array, with rows and columns indexed by $[R]$ and $[C]$, respectively.

Let P and Q be two sets satisfying $|P| = m, |Q| = n, P \cap [R] = \emptyset, Q \cap [C] = \emptyset$.

For each $i \in [s]$, consider an IGBTP$_1(K, |G_i| + w, (r_i + m) \times (c_i + n); w, m \times n)$ $(X_i, \mathcal{A}_i)$, where $X_i = G_i \cup \{\infty_1, \infty_2, \cdots, \infty_w\}$, and whose rows and columns are indexed by $P \cup R_i$ and $Q \cup C_i$, respectively. It can be verified that $(X', \mathcal{A}')$, where

$$X' = X \cup \{\infty_1, \infty_2, \cdots, \infty_w\},$$
$$\mathcal{A}' = \mathcal{A} \cup (\cup_{i=1}^s \mathcal{A}_i),$$

is an IRP$(\sum_{i=1}^s |G_i| + w, K, 1)$.

Arrange the blocks of $(X', \mathcal{A}')$ into an $(R + m') \times (C + n')$ array A, whose rows and columns are indexed by $P \cup [R]$ and $Q \cup [C]$, respectively, such that each block in $\mathcal{A}$ that appears in cell (i, j) of either the FGBTD or the IGBTP, is placed in cell (i, j) of A.

The definition conditions of an FGBTD ensures that no cells are occupied by two blocks. It is also easily checked that every point in X' appears exactly once in each column and either once or twice in each row. In addition, the $m \times n$ subarray

indexed by $P \times Q$ is empty. This gives an $\mathrm{IGBTP}_1(K, \sum_{i=1}^{s} |G_i| + w, (\sum_{i=1}^{s} r_i + m) \times (\sum_{i=1}^{s} c_i + n); w, m \times n)$.

The last statement follows from Proposition 6.3. ☐

Since a GBTD is an instance of GBTP, we have the following recursive construction to produce GBTDs.

COROLLARY 6.5 (FGBTD Construction for GBTD). *Suppose there exists an* $\mathrm{FGBTD}(k, T)$ $(X, \mathcal{G}, \mathcal{A})$, *where* $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$, *and let* $g_i = |G_i|/k$, *for* $i \in [s]$. *If there exists a special* $GBTD_1(k, g_i + 1)$ *for all* $i \in [s]$, *then there exists a special* $GBTD_1(k, \sum_{i=1}^{s} g_i + 1)$.

When the groups are of the same size, we have the following corollary.

COROLLARY 6.6. *If there exists an* $\mathrm{FGBTD}(3, (3g)^t)$ *and a special* $\mathrm{GBTD}_1(3, g+1)$, *then there exists a special* $\mathrm{GBTD}_1(3, gt + 1)$.

For Proposition 6.3 and Corollary 6.5 to be useful, we require large classes of FGBTDs. We give three recursive constructions for FGBTDs next.

6.2. Recursive Constructions for FGBTDs.

PROPOSITION 6.7 (Inflation). *Suppose there exists an* $\mathrm{FGBTD}(k, T)$ *and a* $\mathrm{DRTD}(k, n)$. *Then there exists an* $\mathrm{FGBTD}(k, nT)$.

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be an $\mathrm{FGBTD}(k, T)$ arranged in an $R \times C$ array A, with rows and columns indexed by $[R]$ and $[C]$, respectively. Define

$$X' = X \times [n],$$
$$\mathcal{G}' = \{G \times [n] : G \in \mathcal{G}\},$$

and for each block $A \in \mathcal{A}$, let

$$X_A = A \times [n],$$
$$\mathcal{G}_A = \{\{x\} \times [n] : x \in A\}.$$

and let $(X_A, \mathcal{G}_A, \mathcal{B}_A)$ be a $\mathrm{DRTD}(k, n)$ whose blocks are arranged in an $n \times n$ array with rows and columns both indexed by $[n]$. Let $\mathcal{A}' = \cup_{A \in \mathcal{A}} \mathcal{B}_A$ and the blocks in $\mathcal{A}'$ can be arranged, as follows, in an $Rn \times Cn$ array, whose rows and columns are indexed by $[R] \times n$ and $[C] \times n$, respectively: a block $B \in \mathcal{B}_A$ is placed in cell $((i, a), (j, b))$ if A appears in cell (i, j) of the $\mathrm{FGBTD}(k, T)$ and B appears in cell (a, b) of the $\mathrm{DRTD}(k, n)$. Hence, $(X', \mathcal{G}', \mathcal{A}')$ gives an $\mathrm{FGBTD}(k, nT)$. ☐

Wilson's Fundamental Construction for GDDs [41] can also be modified to construct FGBTDs. Fig. 6.1 describes this construction.

PROPOSITION 6.8 (Fundamental Construction). *Suppose there exists a (master) GDD* $(X, \mathcal{G}, \mathcal{A})$ *of type* T *and let* $w : X \to \mathbb{Z}_{\geq 0}$ *be a weight function. If for each* $A \in \mathcal{A}$, *there exists an (ingredient)* $\mathrm{FGBTD}(k, \langle w(a) : a \in A \rangle)$, *then there exists an* $\mathrm{FGBTD}(k, \langle \sum_{x \in G} w(x) : G \in \mathcal{G} \rangle)$.

Proof. The Fundamental Construction in Fig. 6.1 constructs the desired FGBTD from the master GDD and ingredient FGBTDs. ☐

Proposition 6.8 admits the following specialization.

PROPOSITION 6.9 (FGBTD from Truncated TD). *Let* $s > 0$. *Suppose there exists a* $TD(k + s, m)$, *and* $g_1, g_2, \ldots, g_s$ *are nonnegative integers at most* m. *If there exists*

Input:	(master) GDD $\mathcal{D} = (X, \mathcal{G}, \mathcal{A})$;
	weight function $w \to \mathbb{Z}_{\geq 0}$;
	(ingredient) FGBTD(k, T_A) $\mathcal{D}_A = (X_A, \mathcal{G}_A, \mathcal{B}_A)$ for each $A \in \mathcal{A}$, where
	$T_A = \langle w(x) : x \in A \rangle$,
	$X_A = \cup_{x \in A}(\{x\} \times [w(a)])$,
	$\mathcal{G}_A = \{\{x\} \times [w(x)] : x \in A\}$,
	and the blocks in $\mathcal{B}_A$ are arranged in a $\frac{1}{k}\sum_{x \in A} w(x) \times \frac{1}{k-1}\sum_{x \in A} w(x)$ array,
	whose rows and columns are indexed by $\cup_{x \in A}(\{x\} \times [w(x)/k])$ and
	$\cup_{x \in A}(\{x\} \times [w(x)/(k-1)])$, respectively.
Output:	FGBTD$(k, \langle \sum_{x \in G} w(x) : G \in \mathcal{G} \rangle)$ $\mathcal{D}^* = (X^*, \mathcal{G}^*, \mathcal{A}^*)$, where
	$X^* = \cup_{x \in X}(\{x\} \times [w(x)])$,
	$\mathcal{G}^* = \{\cup_{x \in G}(\{x\} \times [w(x)]) : G \in \mathcal{G}\}$,
	$\mathcal{A}^* = \cup_{A \in \mathcal{A}} \mathcal{B}_A$, and
	the blocks in $\mathcal{A}^*$ are arranged in a $\frac{1}{k}\sum_{x \in X} w(x) \times \frac{1}{k-1}\sum_{x \in X} w(x)$ array,
	whose rows and columns are indexed by $\cup_{x \in X}(\{x\} \times [w(x)/k])$ and
	$\cup_{x \in X}(\{x\} \times [w(x)/(k-1)])$, respectively,
	by placing a block $B \in \mathcal{B}_A$ in cell (i, j) of $\mathcal{D}^*$ if it appears in cell (i, j) of $\mathcal{D}_A$.
Note:	By convention, for $x \in X$, $\{x\} \times [w(x)] = \varnothing$ if $w(x) = 0$.

FIG. 6.1. *Fundamental Construction for FGBTDs*

an FGBTD(k, g^t) *for each* $t \in \{k, k+1, \ldots, k+s\}$, *then there exists an* FGBTD(k, T), *where* $T = (g \cdot m)^k (g \cdot g_1)(g \cdot g_2) \cdots (g \cdot g_s)$.

Proof. For each $i \in [s]$, delete $m - g_i$ points from the ith group of the TD$(k+s, m)$. This results in a $\{k, k+1, \ldots, k+s\}$-GDD of type $m^k g_1 g_2 \cdots g_s$. Use this as the master GDD and apply the fundamental construction with weight function w that assigns weight g to all points. ∎

7. Direct Constructions. This section constructs some small GBTDs and FGBTDs that are required to seed the recursive constructions given in the previous section.

7.1. Direct Constructions for GBTDs. We enforce additional conditions to Definition 4.1, so as to construct the class of GBTDs required to seed the recursive methods.

DEFINITION 7.1 (Special and 3-∗colorable Starter for GBTD$_1(3, m)$). *Let* $\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}$ *be a* $(\Gamma \times [3])$*-GBTD-starter be as defined in Definition 4.1.* $\mathcal{S}$ *is said to be* special *if*

(i) each element in A_0 *appears exactly once in* R*, where* R *is Definition 4.1.*

Furthermore, $\mathcal{S}$ *is said to be* 3-∗colorable with property Π *if each of the blocks in*

$$\{A_\alpha - \alpha : \alpha \in \Gamma\} \text{ and } \{B_t : t \in T\},$$

can be colored with one of three colors so that

(ii) blocks of the same color are pairwise disjoint,

(iii) for each color c*, there exists a point (a* witness *for* c*) that is not contained in any block assigned color* c*.*

PROPOSITION 7.2. *If there exists a special* $(\Gamma \times [3])$*-GBTD-starter, then there exists a special GBTD*$_1(3, m)$*. Similarly, if there exists a* 3-∗*colorable* $(\Gamma \times [3])$*-GBTD-*

starter with property Π*, then there exists a* 3-$*$*colorable* $GBTD_1(3,m)$ *with property* Π.

Proof. Let $\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}$ be a special $(\Gamma \times [3])$-GBTD-starter. By Proposition 4.2, we then have a $\mathrm{GBTD}_1(3,m)$ and condition (i) of 7.1 ensures that the cell $(0,0)$ is special.

On the other hand, let $\mathcal{S}$ be a 3-$*$colorable $(\Gamma \times [3])$-GBTD-starter and let

$$c_i \text{ be the color assigned to } \begin{cases} A_i - i, & \text{if } i \in \Gamma, \\ B_i, & \text{otherwise.} \end{cases}$$

For $\alpha, \beta \in \Gamma$ and $t \in T$, assign the block $A_\alpha + \beta$ color c_α and the block $B_t + \beta$ color c_t. Then conditions (ii) and (iii) of Definition 7.1 ensure that the $\mathrm{GBTD}_1(3,m)$ is 3-$*$colorable with property Π. $\square$

Proposition 7.3. *Let* $q \equiv 1 \bmod 6$*. Then there exists a special* $(\mathbb{F}_q \times [3])$*-GBTD starter that is* 3-$*$*colorable with property* Π.

Proof. Let $s = (q-1)/6$ and ω be a primitive element of $\mathbb{F}_q$. Consider $\gamma \in \mathbb{F}_q$ that satisfies the following conditions (note that ω^{2s} has order three):
(A) $\gamma \notin \{0, -1, -\omega^{2s}, -\omega^{4s}\}$;
(B) $\gamma \notin \left\{ \dfrac{\omega^{2is} - \omega^{t+2js}}{\omega^t - 1} : i \neq j \in [3], t \in [s-1] \right\}$.
The existence of γ is guaranteed since the cardinality of the union of sets in (A) and (B) is at most $4 + 6(s-1) < 6s + 1 = q$.

Let

$$\mathcal{D} = \{\{\omega^{t+2(j-1)s} : j \in [3]\} : t \in \mathbb{Z}_s\}.$$

Wilson [40] showed that the blocks in $\mathcal{D}$ are mutually disjoint and $\Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}$.

In addition, (A) and (B) ensure that for $i \neq i' \in [3]$,

$$\frac{\omega^{2is} + \gamma}{\omega^{2i's} + \gamma} \notin \{0\} \cup \{\omega^t : t \in [s-1]\}.$$

By Definition 4.3, $\{1, \omega^{2s}, \omega^{4s}\} \cup \{\omega\} \cup \{\gamma\}$ forms a $(q,3,1)$-base set for a GBTD-starter. Hence a $(\mathbb{F}_q \times [3])$-GBTD-starter exists.

Define Λ to be $\left\{-\gamma\omega^{t-1+2(j-1)s} : t \in [s], j \in [3]\right\}$ and construct the following $q + 3s = (3q-1)/2$ blocks. For $\alpha \in \mathbb{F}_q$, let

$$A_\alpha = \begin{cases} \left\{ \left(\omega^{t-1+2(j-1)s}\right)_i : j \in [3] \right\}, & \text{if } \alpha = -\gamma\omega^{t-1+2(i-1)s} \\ & \text{where } (t,i) \in [s] \times [3], \\ \left\{ \left(-\frac{\alpha}{\gamma}\omega^{2(i-1)s}\right)_i : i \in [3]) \right\}, & \text{otherwise.} \end{cases}$$

For $(t,j) \in [s] \times [3]$, let

$$B_{(t,j)} = \left\{ \left(\omega^{t-1+2(j-1)s}\left(\omega^{2(i-1)s} + \gamma\right)\right)_i : i \in [3] \right\}$$

Let $\mathcal{S} = \{A_\alpha : \alpha \in \mathbb{F}_q\} \cup \{B_{(t,j)} : (t,j) \in [s] \times [3]\}$. Then by the proof of Proposition 4.4, $\mathcal{S}$ is an $(\mathbb{F}_q \times [3])$-GBTD-starter.

Next, observe that $A_0 = \{(0,i) : i \in [3]\}$. By Claim 4.1, to establish condition (i) of Definition 7.1, it suffices to show that $0_i \notin A_\alpha - \alpha$ for $\alpha \in \Lambda$ and $i \in [3]$. Suppose otherwise. Then there exists $(t,j) \in \mathbb{Z}_s \times [k]$ and $i \in [3]$ such that

$$(\omega^{(j-1)s} + \gamma)\omega^{t+(i-1)s} = 0,$$

contradicting (A).

Finally, we exhibit that $\mathcal{S}$ is 3-$*$colorable with property Π by assigning the block A_0 color $\clubsuit$, the blocks $A_\alpha - \alpha$ for $\alpha \notin \Lambda$ and B_t for $t \in T$ color $\heartsuit$ and the blocks $A_\alpha - \alpha$ for $\alpha \in \Lambda$ color $\diamondsuit$. Then this assignment satisfies condition (i) of Definition 7.1. In addition, 0_1 is a witness for both $\heartsuit$ and $\diamondsuit$ and α_1 is a witness for $\clubsuit$ for some $\alpha \neq 0$, satisfying condition (ii) of Definition 7.1. $\square$

Corollary 7.4. *Let* $q \equiv 1 \bmod 6$. *Then a* 3-$*$*colorable* $\mathrm{GBTD}_1(3,m)$ *with property* Π *exists.*

Proof. This follows from Proposition 4.2 and Proposition 7.3. $\square$

Corollary 7.5. *A special* $\mathrm{GBTD}_1(3,m)$ *exists for* $m \in \{1,17,29,35,47,53,55\}$, *a* 3-$*$*colorable special* $\mathrm{GBTD}_1(3,m)$ *with property* Π *for* $m \in \{9,11,23\}$ *and a* 3-$*$*colorable* $\mathrm{RBIBD}(15,3,1)$ *with property* Π.

Proof. A special $\mathrm{GBTD}_1(3,1)$ exists trivially. On the other hand, a 3-$*$colorable special $\mathrm{GBTD}_1(3,9)$ with property Π is given by Example 5.4, and a 3-$*$colorable $\mathrm{RBIBD}(15,3,1)$ with property Π is given by Example 5.1.

For $m \in \{11,17,23,29,35,47,53,55\}$, apply Proposition 4.2 with special $(\mathbb{Z}_m \times [3])$-GBTD-starters and 3-$*$colorable special $(\mathbb{Z}_m \times [3])$-GBTD-starters with property Π given in [10]. $\square$

7.2. Direct Constructions for an IGBTP$_1(\{2,3^*\}, 2m+w, (m+(w-1)/2) \times (2m+w-4); w, (w-1)/2 \times (w-4))$. As with GBTDs, we use a set of starters to construct GBTPs. To construct this starters, we need the notion of infinity elements.

Given an abelian group Γ, we augment the point set with *infinity* elements, denoted by ∞_i where i belongs to some index set I. The infinity elements are fixed under addition by elements in Γ. That is, $\infty_i + \gamma = \infty_i$ for $\gamma \in \Gamma$. Let w be a positive integer and $W_w = \{\infty_i : i \in [w]\}$. So, given a block $A \subset \Gamma \cup W_w$ and $\gamma \in \Gamma$, $A + \gamma = \{a + \gamma : a \in A \setminus W_w\} \cup (A \cap W_w)$.

We also extend the definition of difference lists. For a set system $(\Gamma \cup W_w, \mathcal{S})$, then the difference list of $\mathcal{S}$ is given by the multiset

$$\Delta\mathcal{S} = \langle x - y : x, y \in A \setminus W_w, x \neq y, A \in \mathcal{S}\rangle.$$

Definition 7.6. *Let* m *be an odd integer with* $m \geq 11$ *Let* $(\mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w, \mathcal{S})$ *be a* $\{2,3\}$*-uniform set system of size* $w - 3 + m$, *where*

$$\mathcal{S} = \{A_i : i \in [(w-5)/2]\} \cup \{B_i : i \in [(w-1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\}.$$

satisfying $|A_i| = 2$ *for* $i \in [(w-5)/2]$, $|B_i| = 2$ *for* $i \in [(w-1)/2]$, $|C_0| = 3$, *and* $|C_i| = 2$ *for* $i \in \mathbb{Z}_m \setminus \{0\}$.

$\mathcal{S}$ *is called a* $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-IGBTP-starter *if the following conditions hold:*

(i) $\Delta\mathcal{S} = \mathbb{Z}_m \times \mathbb{Z}_2 \setminus \{0_0, 0_1\}$,

(ii) $\{j : a_j \in A_i\} = \mathbb{Z}_2$ *for* $i \in [(w-5)/2]$,

(iii) $\{B_i : i \in [(w-1)/2]\} \cup \{C_j : j \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$,

(iv) $|C_i \cap W_w| \leq 1$ *for* $i \in \mathbb{Z}_m$,

(v) each element in $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$ *appears either once or twice in the multiset*

$$R = \{0_0, 0_1\} \cup \left(\bigcup_{\substack{i \in [(w-5)/2] \\ j \in \mathbb{Z}_2}} A_i + 0_j \right) \cup \left(\bigcup_{i_j \in \mathbb{Z}_m \times \mathbb{Z}_2} C_i - i_j \right).$$

$$
\begin{array}{|c|c|c|}
\hline
\mathsf{W} & \mathsf{B} & \mathsf{B}+0_1 \\
\hline
\mathsf{A} & \mathsf{C} & \mathsf{C}+0_1 \\
\hline
\end{array}
$$

where W is a $(w-1)/2 \times (w-4)$ empty array, A is an $m \times (w-4)$ array,

$$
\begin{array}{|c|ccccccc|}
\hline
\{0_0, 0_1\} & A_1 & A_1+0_1 & A_2 & A_2+0_1 & \cdots & A_{(w-5)/2} & A_{(w-5)/2}+0_1 \\
\{1_0, 1_1\} & A_1+1_0 & A_1+1_1 & A_2+1_0 & A_2+1_1 & \cdots & A_{(w-5)/2}+1_0 & A_{(w-5)/2}+1_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\{(m-1)_0, (m-1)_1\} & A_1-1_0 & A_1-1_1 & A_2-1_0 & A_2-1_1 & \cdots & A_{(w-5)/2}-1_0 & A_{(w-5)/2}-1_1 \\
\hline
\end{array},
$$

B and C are the following $(w-1)/2 \times m$ and $m \times m$ arrays,

$$
\begin{array}{|cccc|}
\hline
B_1 & B_1+1_0 & \cdots & B_1-1_0 \\
B_2 & B_1+1_0 & \cdots & B_1-1_0 \\
\vdots & \vdots & \ddots & \vdots \\
B_{(w-1)/2} & B_{(w-1)/2}+1_0 & \cdots & B_{(w-1)/2}-1_0 \\
\hline
\end{array},
\quad
\begin{array}{|cccc|}
\hline
C_0 & C_{m-1}+1_0 & \cdots & C_1-1_0 \\
C_1 & C_0+1_0 & \cdots & C_2-1_0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{m-1} & C_{m-2}+1_0 & \cdots & C_0-1_0 \\
\hline
\end{array}.
$$

FIG. 7.1. *An* IGBTP$_1(\{2,3^*\}, 2m+w, (m+(w-1)/2) \times (2m+w-4); w, (w-1)/2 \times (w-4))$ *from a* $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$*-GBTP-starter.*

$$
\begin{array}{|c|c|c|c|c|}
\hline
\mathsf{W} & \mathsf{B} & \mathsf{B}+0_1 & \mathsf{B}+0_2 & \mathsf{B}+0_3 \\
\hline
 & \mathsf{C} & \mathsf{D}+0_1 & \mathsf{C}+0_2 & \mathsf{D}+0_3 \\
\cline{2-5}
\mathsf{A} & \mathsf{D} & \mathsf{C}+0_1 & \mathsf{D}+0_2 & \mathsf{C}+0_3 \\
\hline
\end{array}
$$

where W is a 4×5 empty array, A is a $2m \times 5$ array,

$$
\begin{array}{|ccc|cc|}
\hline
\{0_0, 0_1\} & \{x_0, x_2\} & \{y_0, y_3\} & A & A+0_2 \\
\{1_0, 1_1\} & \{(x+1)_0, x_2\} & \{(y+1)_0, (y+1)_3\} & A+1_0 & A+1_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\{(m-1)_0, (m-1)_1\} & \{(x-1)_0, x_2\} & \{(y-1)_0, (y-1)_3\} & A+(m-1)_0 & A+(m-1)_2 \\
\hline
\{0_2, 0_3\} & \{x_1, x_3\} & \{y_1, y_2\} & A+0_1 & A+0_3 \\
\{1_2, 1_3\} & \{(x+1)_1, x_3\} & \{(y+1)_1, (y+1)_2\} & A+1_1 & A+1_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\{(m-1)_2, (m-1)_3\} & \{(x-1)_1, x_3\} & \{(y-1)_1, (y-1)_2\} & A+(m-1)_1 & A+(m-1)_3 \\
\hline
\end{array},
$$

B, C and D are the following $4 \times m$, $m \times m$ and $m \times m$ arrays respectively,

$$
\begin{array}{|cccc|}
\hline
B_1 & B_1+1_0 & \cdots & B_1-1_0 \\
B_2 & B_2+1_0 & \cdots & B_2-1_0 \\
B_3 & B_3+1_0 & \cdots & B_3-1_0 \\
B_4 & B_4+1_0 & \cdots & B_4-1_0 \\
\hline
\end{array},
\quad
\begin{array}{|cccc|}
\hline
C_0 & C_{m-1}+1_0 & \cdots & C_1-1_0 \\
C_1 & C_0+1_0 & \cdots & C_2-1_0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{m-1} & C_{m-2}+1_0 & \cdots & C_0-1_0 \\
\hline
\end{array},
\quad
\begin{array}{|cccc|}
\hline
D_0 & D_{m-1}+1_0 & \cdots & D_1-1_0 \\
D_1 & D_0+1_0 & \cdots & D_2-1_0 \\
\vdots & \vdots & \ddots & \vdots \\
D_{m-1} & D_{m-2}+1_0 & \cdots & D_0-1_0 \\
\hline
\end{array}.
$$

FIG. 7.2. *An* IGBTP$_1(\{2,3^*\}, 4m+9, (2m+4) \times (4m+5); 9, 4 \times 5)$ *from a* $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$*-GBTP-starter.*

PROPOSITION 7.7. *If there exists a* $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$*-IGBTP-starter,then there exists an IGBTP*$_1(\{2,3^*\}, 2m+w, (m+(w-1)/2)\times(2m+w-4); w, (w-1)/2\times(w-4))$.

Proof. Let

$$\begin{aligned} X &= \mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w, \\ \mathcal{A} &= \{S + j : S \in \mathcal{S} \text{ and } j \in \mathbb{Z}_m \times \mathbb{Z}_2\} \cup \{\{i_0, i_1\} : i \in \mathbb{Z}_m\}. \end{aligned}$$

Then $(X, W_w, \mathcal{A})$ is an IRP$(2m + w, K, 1; w)$, whose blocks can be arranged in an $(m + (w - 1)/2) \times (2m + w - 4)$ array as in Figure 7.2. We index the rows by $[(w - 1)/2] \cup \mathbb{Z}_m$ and the columns by $[w - 4] \cup (\mathbb{Z}_m \times \mathbb{Z}_2)$.

First, check that the cell (r, c) is empty for $(r, c) \in [(w - 1)/2] \times [w - 4]$.

For $j \in [w-4]$, the set of blocks occupying column j is $\mathbb{Z}_m \times \mathbb{Z}_2$ by condition (ii) of Definition 7.6. For $j \in \mathbb{Z}_m \times \mathbb{Z}_2$, first observe that the set of the blocks occupying the column 0_0 by condition (iii) of Definition 7.6 is $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$. Since the blocks of column j are translates (by j) of the blocks in column 0_0, the union of the blocks in column j is also $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$.

For $i \in [(w - 1)/2]$, each element in $\mathbb{Z}_m \times \mathbb{Z}_2$ appears exactly twice in row i by construction. For $i \in \mathbb{Z}_m$, let R_i denote the multiset containing all the points appearing in the blocks of row i. Then $R_0 = R$ and $R_i = R_0 + i_0$, for all $i \in \mathbb{Z}_m$. Hence, it suffices each element in X appears either once or twice in R, which follows immediately from conditions (v) in Definition 7.6. □

DEFINITION 7.8. *Let m be an odd integer with $m \geq 11$. Let $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9, \mathcal{S})$ be a $\{1, 2, 3\}$-uniform set system of size $7 + 2m$, where*

$$\mathcal{S} = \{x_0\} \cup \{y_0\} \cup A \cup \{B_i : i \in [4]\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\}.$$

satisfying $|A| = 2$, $|B_i| = 2$ for $i \in [4]$, $|C_0| = 3$, $|C_i| = 2$ for $i \in \mathbb{Z}_m \setminus \{0\}$ and $|D_i| = 2$ for $i \in \mathbb{Z}_m$.

$\mathcal{S}$ is called a $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-IGBTP-starter *if the following conditions hold:*

(i) $\Delta\mathcal{S} = (\mathbb{Z}_m \times \mathbb{Z}_4) \setminus \{0_0, 0_1, 0_2, 0_3\}$,
(ii) $\{j : a_j \in A\} = \{0, 2\}$,
(iii) $\{B_i : i \in [(w - 1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$,
(iv) $|C_i \cap W_9| \leq 1$ *and* $|D_i \cap W_9| \leq 1$ *for* $i \in \mathbb{Z}_m$,
(v) each element in $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$ *appears either once or twice in the multisets*

$$\begin{aligned} R_\circ &= \{0_0, 0_1, x_0, x_2, y_0, y_3\} \cup A \cup A + 0_2 \\ &\quad \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{0,2\}} C_i - i_j \right) \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{1,3\}} D_i - i_j \right), \\ R_\bullet &= \{0_2, 0_3, x_1, x_3, y_1, y_2\} \cup A + 0_1 \cup A + 0_3 \\ &\quad \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{1,3\}} C_i - i_j \right) \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{0,2\}} D_i - i_j \right). \end{aligned}$$

PROPOSITION 7.9. *If there exists a* $(\mathbb{Z}_m \times \mathbb{Z}_4 \cup W_9)$*-IGBTP-starter, then there exists an IGBTP*$_1(\{2, 3^*\}, 4m + 9, (2m + 4) \times (4m + 5); 9, 4 \times 5)$.

Proof. Let

$$
\begin{aligned}
X &= (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9, \\
\mathcal{A} &= \{S + j : S \in \mathcal{S},\ |S| \neq 1,\ j \in \mathbb{Z}_m \times \mathbb{Z}_2\} \cup \{\{i_0, i_1\} : i \in \mathbb{Z}_m\} \cup \{\{i_2, i_3\} : i \in \mathbb{Z}_m\} \\
&\quad \cup \{\{(x+i)_0, (x+i)_2\} : i \in \mathbb{Z}_m\} \cup \{\{(x+i)_1, (x+i)_3\} : i \in \mathbb{Z}_m\} \\
&\quad \cup \{\{(y+i)_0, (y+i)_3\} : i \in \mathbb{Z}_m\} \cup \{\{(y+i)_1, (y+i)_2\} : i \in \mathbb{Z}_m\}.
\end{aligned}
$$

Then $(X, W_9, \mathcal{A})$ is an IRP$(4m+9, K, 1; 9)$, whose blocks can be arranged in a $(2m+4) \times (4m+5)$ array as in Figure 7.1. We index the rows by $[4] \cup (\mathbb{Z}_m \times \{\circ, \bullet\})$ and the columns by $[5] \cup (\mathbb{Z}_m \times \mathbb{Z}_4)$.

First, check that the cell (r, c) is empty for $(r, c) \in [4] \times [5]$.

For $j \in [5]$, the set of blocks occupying column j is $\mathbb{Z}_m \times \mathbb{Z}_4$ by condition (ii) of Definition 7.8. For $j \in \mathbb{Z}_m \times \mathbb{Z}_4$, first observe that the set of the blocks occupying the column 0_0 by condition (iii) of Definition 7.8 is $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$. Since the blocks of column j are translates (by j) of the blocks in column 0_0, the union of the blocks in column j is also $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$.

For $i \in [4]$, each element in $\mathbb{Z}_m \times \mathbb{Z}_4$ appears exactly twice in row i by construction. For $(i, *) \in \mathbb{Z}_m \times \{\circ, \bullet\}$, let $R_{(i,*)}$ denote the multiset containing all the points appearing in the blocks of row $(i, *)$. Then $R_{(0,*)} = R_*$ and $R_{(i,*)} = R_{(0,*)} + i_0$, for all $i \in \mathbb{Z}_m$. Hence, it suffices each element in X appears either once or twice in R_*, which follows immediately from conditions (v) in Definition 7.8. $\square$

Corollary 7.10. *An IGBTP*$_1(\{2, 3^*\}, 2m+9, (m+4) \times (2m+5); 9, 4 \times 5)$ *exists for* $m \in \{s : 10 \leq s \leq 45\} \cup \{47, 49, 53, 57, 77\} \setminus \{16, 20, 24, 28, 36, 40, 44\}$, *and an IGBTP*$_1(\{2, 3^*\}, 2m+11, (m+5) \times (2m+7); 11, 5 \times 7)$ *exists for* $m \in \{15, 19, 23, 27, 31, 35, 45, 49\}$.

Proof. The required $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_9)$-IGBTP-starter for $m \in \{s : 11 \leq s \leq 49, s \text{ odd}\} \cup \{53, 57, 77\}$ and $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-IGBTP starter for $m \in \{s : 5 \leq s \leq 21, s \text{ odd}\}$ is given in [10] and we apply Proposition 7.7 and Proposition 7.9 to obtain the corresponding IGBTP.

Similarly, to construct an IGBTP$_1(\{2, 3^*\}, 2m+11, (m+5) \times (2m+7); 11, 5 \times 7)$ for $m \in \{15, 19, 23, 27, 31, 35, 45, 49\}$, we apply Proposition 7.7 to $(\mathbb{Z}_m \times \mathbb{Z}_2 \cup W_{11})$-IGBTP starters listed in [10].

It remains to construct an IGBTP$_1(\{2, 3^*\}, 33, 16 \times 29; 9, 4 \times 5)$. Consider $((\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W_9, \mathcal{S})$, a $\{2, 3\}$-uniform set system of size 36, where $\mathcal{S}$ comprise the blocks below:

$$
\begin{array}{llll}
A_1 = \{1_0, 1_2\} & A_2 = \{1_1, 1_5\} & A_3 = \{0_0, 0_4\} & A_4 = \{1_3, 1_6\} \\
A_5 = \{0_3, 0_5\} & A_6 = \{1_1, 1_3\} & A_7 = \{1_4, 1_7\} & A_8 = \{0_1, 0_6\} \\
A_9 = \{0_0, 0_5\} & A_{10} = \{0_2, 0_4\} & A_{11} = \{1_4, 1_6\} & A_{12} = \{1_0, 1_3\} \\
A_{13} = \{0_2, 0_5\} & A_{14} = \{1_2, 1_7\} & A_{15} = \{0_1, 0_7\} & A_{16} = \{1_5, 1_7\} \\
A_{17} = \{0_2, 0_6\} & A_{18} = \{0_3, 0_7\} & A_{19} = \{1_1, 1_4\} & A_{20} = \{1_0, 1_6\} \\
B_1 = \{0_0, 0_1\} & B_2 = \{0_5, 1_5\} & B_3 = \{1_1, 2_4\} & B_4 = \{0_7, 1_3\} \\
C_0^1 = \{1_0, 2_1, 2_6\} & C_1^1 = \{1_0, 2_1\} & C_2^1 = \{1_0, 2_1\} & \\
C_0^2 = \{0_2, \infty_1\} & C_1^2 = \{0_4, \infty_2\} & C_2^2 = \{1_2, \infty_3\} & \\
C_0^3 = \{2_0, \infty_4\} & C_1^3 = \{2_3, \infty_5\} & C_2^3 = \{1_6, \infty_6\} & \\
C_0^4 = \{2_7, \infty_7\} & C_1^4 = \{2_2, \infty_8\} & C_2^4 = \{2_5, \infty_9\}. &
\end{array}
$$

W	B	$\mathsf{B}+0_1$	$\mathsf{B}+0_2$	$\mathsf{B}+0_3$	$\mathsf{B}+0_4$	$\mathsf{B}+0_5$	$\mathsf{B}+0_6$	$\mathsf{B}+0_7$
A	C_1	C_4+0_1	C_3+0_2	C_2+0_3	C_1+0_4	C_4+0_5	C_3+0_6	C_2+0_7
	C_2	C_1+0_1	C_4+0_2	C_3+0_3	C_2+0_4	C_1+0_5	C_4+0_6	C_3+0_7
	C_3	C_2+0_1	C_1+0_2	C_4+0_3	C_3+0_4	C_2+0_5	C_1+0_6	C_4+0_7
	C_4	C_3+0_1	C_2+0_2	C_1+0_3	C_4+0_4	C_3+0_5	C_2+0_6	C_1+0_7

where W is a 4×5 empty array, A is a 12×5 array,

$$\begin{array}{|ccccc|}\hline
A_1 & A_2 & A_3 & A_4 & A_5 \\
A_1+1_0 & A_2+1_0 & A_3+1_0 & A_4+1_0 & A_5+1_0 \\
A_1+2_0 & A_2+2_0 & A_3+2_0 & A_4+2_0 & A_5+2_0 \\
A_6 & A_7 & A_8 & A_9 & A_{10} \\
A_6+1_0 & A_7+1_0 & A_8+1_0 & A_9+1_0 & A_{10}+1_0 \\
A_6+2_0 & A_7+2_0 & A_8+2_0 & A_9+2_0 & A_{10}+2_0 \\
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{11}+1_0 & A_{12}+1_0 & A_{13}+1_0 & A_{14}+1_0 & A_{15}+1_0 \\
A_{11}+2_0 & A_{12}+2_0 & A_{13}+2_0 & A_{14}+2_0 & A_{15}+2_0 \\
A_{16} & A_{17} & A_{18} & A_{19} & A_{20} \\
A_{16}+1_0 & A_{17}+1_0 & A_{18}+1_0 & A_{19}+1_0 & A_{20}+1_0 \\
A_{16}+2_0 & A_{17}+2_0 & A_{18}+2_0 & A_{19}+2_0 & A_{20}+2_0 \\
\hline\end{array},$$

B is a 4×3 array,

$$\begin{array}{|ccc|}\hline
B_1 & B_1+1_0 & B_1+2_0 \\
B_2 & B_2+1_0 & B_2+2_0 \\
B_3 & B_3+1_0 & B_3+2_0 \\
B_4 & B_4+1_0 & B_4+2_0 \\
\hline\end{array},$$

C_i for $i \in [4]$ is a 3×3 array,

$$\begin{array}{|ccc|}\hline
C_0^i & C_2^i+1_0 & C_1^i+2_0 \\
C_1^i & C_0^i+1_0 & C_2^i+2_0 \\
C_2^i & C_1^i+1_0 & C_0^i+2_0 \\
\hline\end{array}.$$

FIG. 7.3. *An* $\mathrm{IGBTP}_1(\{2,3^*\},33,16\times 29;9,4\times 5)$.

Let

$$X = (\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W$$
$$\mathcal{A} = \{S + j : S \in \mathcal{S}, j \in \mathbb{Z}_3 \times \mathbb{Z}_8\}.$$

Then $(X, W, \mathcal{A})$ is an $\mathrm{IRP}(33, \{2, 3^*\}, 1; 9)$, whose blocks can be arranged in a 16×29 array as in Figure 7.3. It can be readily verified that this arrangement results in an $\mathrm{IGBTP}_1(\{2, 3^*\}, 33, 16 \times 29; 9, 4 \times 5)$. $\square$

7.3. Direct Constructions for FGBTDs.

—	—	{2,7}	{6,3}	{7,1}	{3,5}	{5,6}	{1,2}
{2,3}	{6,7}	—	—	{3,0}	{7,4}	{0,2}	{4,6}
{5,7}	{1,3}	{3,4}	{7,0}	—	—	{4,1}	{0,5}
{1,6}	{5,2}	{6,0}	{2,4}	{4,5}	{0,1}	—	—

FIG. 7.4. *An* $\mathrm{FGBTD}_1(2, 2^4)$ $(X, \mathcal{G}, \mathcal{A})$*, where* $X = \mathbb{Z}_8$ *and* $\mathcal{G} = \{\{i, 4+i\} : i \in \mathbb{Z}_4\}$.

—	—	{7,9}	{2,4}	{3,4}	{8,9}	{6,2}	{1,7}	{1,8}	{6,3}
{7,4}	{2,9}	—	—	{8,0}	{3,5}	{4,5}	{9,0}	{7,3}	{2,8}
{3,9}	{8,4}	{8,5}	{3,0}	—	—	{9,1}	{4,6}	{5,6}	{0,1}
{1,2}	{6,7}	{4,0}	{9,5}	{9,6}	{4,1}	—	—	{0,2}	{5,7}
{6,8}	{1,3}	{2,3}	{7,8}	{5,1}	{0,6}	{0,7}	{5,2}	—	—

FIG. 7.5. *An* $\mathrm{FGBTD}_1(2, 2^5)$ $(X, \mathcal{G}, \mathcal{A})$*, where* $X = \mathbb{Z}_{10}$ *and* $\mathcal{G} = \{\{i, 5+i\} : i \in \mathbb{Z}_5\}$.

LEMMA 7.11. *There exists an FGBTD*$(2, 2^t)$ *for* $t \in \{4, 5\}$.

Proof. The desired FGBTDs are given in Fig. 7.4 and Fig. 7.5. ☐

DEFINITION 7.12. *Let* t *be a positive integer, and let* $I = [t-1] \times [2]$. *Let* $(\mathbb{Z}_{3t} \times [2], \mathcal{S})$ *be a* 3*-uniform set system of size* $2(t-1)$*, where* $\mathcal{S} = \{A_i : i \in I\}$. $\mathcal{S}$ *is called a* $(\mathbb{Z}_{3t} \times [2])$-FGBTD-starter *if the following conditions hold:*

(i) $\Delta_{ij}\mathcal{S} = \mathbb{Z}_{3t} \setminus \{0, t, 2t\}$ *for* $i, j \in [2]$,

(ii) $\cup_{i \in I} A_i = (\mathbb{Z}_{3t} \setminus \{0, t, 2t\}) \times [2]$,

(iii) for $j \in [2]$*, each element in* $(\mathbb{Z}_t \setminus \{0\}) \times [2]$ *appears either once or twice in the multiset*

$$R_j = \bigcup_{i=1}^{t-1} A_{(i,j)} - i \bmod t,$$

(iv) $r \in (\mathbb{Z}_t \setminus \{0\}) \times [2]$ *for each* $r \in R_1 \cup R_2$.

PROPOSITION 7.13. *If there exists a* $(\mathbb{Z}_{3t} \times [2], 6^t)$*-FGBTD-starter, then there exists an* $\mathrm{FGBTD}(3, 6^t)$.

Proof. Let

$$\begin{aligned}
X &= \mathbb{Z}_{3t} \times [2], \\
\mathcal{G} &= \{G_i = \{i_1, (t+i)_1, (2t+i)_1, i_2, (t+i)_2, (2t+i)_2\} : i \in \mathbb{Z}_t\}, \\
\mathcal{A} &= \{A_i + j : i \in I \text{ and } j \in \mathbb{Z}_{3t}\}.
\end{aligned}$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a $\{3\}$-GDD of type 6^t, whose blocks can be arranged in a $2t \times 3t$ array, with rows and columns indexed by $\mathbb{Z}_t \times [2]$ and $\mathbb{Z}_{3t}$, respectively, as follows: the block $A_{(i,j)} + k$ is placed in cell $((i+k, j), k)$.

The set of blocks occupying column zero are $\{A_i : i \in I\}$ and by condition (ii) of Definition 7.12, $\bigcup_{i \in I} A_i = X \setminus G_0$. For other $j \in \mathbb{Z}_{3t}$, observe that the blocks occupying column j are translates (by j) of the blocks in column zero, and hence the union of the blocks in column j is $X \setminus G_{j'}$, where $j' \equiv j \bmod t$.

For $(i, j) \in \mathbb{Z}_t \times [2]$, let $R_{(i,j)}$ denote the multiset containing all the points appearing in the blocks of row (i, j). Then $R_{(i,j)} = R_{(0,j)} + i$, for all $i \in \mathbb{Z}_t$. Hence, it suffices to check that each element of $X \setminus G_0$ appears either once or twice in $R_{(0,j)}$ and the elements of $R_{(0,j)}$ belong to $X \setminus G_0$ for $j \in [2]$. This, however, follows immediately from conditions (iii) and (iv) in Definition 7.12, since $R_{(0,j)} = R_j \cup (R_j + t) \cup (R_j + 2t)$ for $j \in [2]$. ☐

Corollary 7.14. *There exist an* FGBTD$(3, 6^t)$ *for all* $t \in \{5, 6, 7, 8\}$*, an* FGBTD$(3, 24^t)$ *for all* $t \in \{5, 8\}$ *and an* FGBTD$(3, 30^t)$ *for all* $t \in \{5, 7\}$*.*

Proof. An FGBTD$_1(3, 6^6)$ is given by Example 5.5. An FGBTD$(3, 6^t)$ for $t \in \{5, 7\}$ exists by applying Proposition 7.13 with FGBTD-starters given in [10].

The existence of an FGBTD$(3, 24^t)$, $t \in \{5, 8\}$ follows by applying Proposition 6.7 with an FGBTD$(3, 6^t)$ (constructed in this proof) and a DRTD$(3, 4)$, whose existence is provided by Corollary 5.11. The existence of an FGBTD$(3, 30^t)$, $t \in \{5, 7\}$ follows by applying Proposition 6.7 similarly.

To prove the existence of an FGBTD$(3, 6^8)$, consider $(\mathbb{Z}_{48}, \mathcal{S})$, a $\{3\}$-uniform set system of size 7, where $\mathcal{S}$ comprise the blocks below:

$A_1 = \{2, 3, 5\}$ $A_2 = \{4, 14, 31\}$ $A_3 = \{9, 22, 45\}$
$A_4 = \{15, 34, 43\}$ $A_5 = \{20, 35, 42\}$ $A_6 = \{13, 17, 47\}$
$A_7 = \{1, 6, 12\}$.

Observe that $\mathcal{S}$ satisfies the following conditions:

(i) $\Delta\mathcal{S} = \mathbb{Z}_{48} \setminus \{0, 8, 16, 24, 32, 40\}$,
(ii) $\cup_{i\in[7]} A_i \bmod 24 = \mathbb{Z}_{24} \setminus \{0, 8, 16\}$,
(iii) each element in $\mathbb{Z}_{16} \setminus \{0, 8\}$ appears either once or twice in the multiset

$$R = \bigcup_{i\in[7]} A_i - i \bmod 16,$$

(iv) $r \in \mathbb{Z}_{16} \setminus \{0, 8\}$ for each $r \in R$.

Further, let

$$\begin{aligned} X &= \mathbb{Z}_{48} \\ \mathcal{G} &= \{\{i + 8k : k \in \mathbb{Z}_6\} : i \in \mathbb{Z}_8\}, \\ \mathcal{A} &= \{A_i + j : i \in [7] \text{ and } j \in \mathbb{Z}_{48}\}. \end{aligned}$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a $\{3\}$-GDD of type 6^8, whose blocks can be arranged in a 16×24 array, with rows and columns are indexed by $\mathbb{Z}_{16}$ and $\mathbb{Z}_{24}$, respectively, as follows: the block $A_i + j$ is placed in cell $(i + j, j)$. This array can be verified to be an FGBTD$(3, 6^8)$. □

8. Existence of GBTDs and GBTPs. In this section, we apply recursive constructions in Section 6 with small designs directly constructed in Section 7 to completely settle the existence of GBTD$_1(3, m)$ and GBTP$_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$.

8.1. Existence of GBTD$_1(3, m)$.

Lemma 8.1. *A special* GBTD$_1(3, 3^r q)$ *exists for all* $r \geq 0$ *and* $q \in Q$*, where* $Q = \{q : q \equiv 1 \bmod 6$ *is a prime power*$\} \cup \{5, 9, 11, 23\}$*, except when* $(r, q) = (0, 5)$*.*

Proof. Existence of a special GBTD$_1(3, q)$ for all $q \in Q \setminus \{5\}$ is provided by Corollary 7.4 and 7.5. These GBTDs are all 3-$*$colorable with property Π. The lemma then follows by considering these GBTDs as RBIBDs and applying Corollary 6.2. □

Lemma 8.2. *Let* $s \in [2]$ *and suppose there exists a* TD$(5 + s, n)$*. If* $0 \leq g_i \leq n$*,* $i \in [s]$ *and that there exists a special* GBTD$_1(3, m)$ *for all* $m \in \{2n + 1\} \cup \{2g_i + 1 : i \in [s]\}$*, then there exists a special* GBTD$_1(3, 10n + 1 + 2\sum_{i=1}^{s} g_i)$*.*

Proof. By Corollary 7.14, there exists an FGBTD$(3, 6^t)$ for all $t \in \{5, 6, 7\}$. By Proposition 6.9, there exists an FGBTD$(3, (6n)^5(6g_1) \cdots (6g_s))$. Now apply Corollary 6.5 to obtain a special GBTD$_1(3, 10n + 1 + 2\sum_{i=1}^{s} g_i)$. □

Table 8.1
Existence of special $\mathrm{GBTD}_1(3,m)$

Authority	m
Corollary 7.5	9, 11, 17, 23, 29, 35, 47, 53, 55
Lemma 8.1	7, 13, 15, 19, 21, 25, 27, 31, 33, 37, 39, 43, 45, 49, 57, 61, 63, 67, 69, 73, 75
Corollary 6.6 with (g,t) in $\{(8,5),(5,10),(8,8),(7,10)\}$	41, 51, 65, 71
Lemma 8.2 with $n=5$, $g_1=4$	59
Lemma 8.2 with $n=7$, $g_1,g_2\in\{0\}\cup\{t:3\le t\le 7\}$	$\{s:77\le s\le 95, s \text{ odd}\}$

Table 8.2
Existence of $IGBTP_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);4\times 5)$

Authority	m
Corollary 7.10	$\{s : 10 \le s \le 57\}\setminus\{16,20,24,28,32,36,40,44,48,50,52,54,55,56\}$
Lemma 8.4 with $(n,g)\in\{(10,0),(11,0),(12,0),(13,0),(11,10),(11,11),(14,0)\}$	40, 44, 48, 52, 54, 55, 56

Lemma 8.3. *A special* $\mathrm{GBTD}_1(3,m)$ *exists for odd* $m\ge 7$.

Proof. First, a special $\mathrm{GBTD}_1(3,m)$ can be constructed for odd m, $7\le m\le 95$. Details are provided in Table 8.1.

We then prove the lemma by induction on $m\ge 97$.

Let $E=\{t:t\ge 9\}\setminus\{10,14,15,18,20,22,26,30,34,38,46,60\}$. By Theorem 5.8, a $\mathrm{TD}(7,n)$ exists for any $n\in E$. If there exists a special $\mathrm{GBTD}_1(3,m')$ for odd m', $7\le m'\le 2n+1$, then apply Lemma 8.2 with $3\le g_1,g_2\le n$ to obtain a special $\mathrm{GBTD}_1(3,m)$ for odd m, $10n+7\le m\le 14n+1$.

Hence, take $n=9$ to obtain a special $\mathrm{GBTD}_1(3,97)$.

Suppose there exists a $\mathrm{GBTD}_1(3,m')$ for all odd $m'<m$. Then there exists $n\in E$ with $10n+7\le m\le 14n+1$. Suppose otherwise. Then there exists $n_1\in E$ such that $14n_1+1<10n_2+7$ for all $n_2>n_1$ and $n_2\in E$. This, together with the fact that $n_1\ge 9$, implies that $n_2-n_1>3$ for all $n_2\in E$ and $n_2>n_1$. However, a quick check on E gives a contradiction.

Since $n\in E$ and there exists a special $\mathrm{GBTD}_1(3,m')$ for all $m'\le 2n+1<10n+7\le m$ (induction hypothesis), there exists a special $\mathrm{GBTD}_1(3,m)$ and induction is complete. ∎

Lemma 8.3 shows that a $\mathrm{GBTD}_1(3,m)$ exists for all odd $m\ne 3,5$. Theorem 2.3(v) now follows.

8.2. Existence of $\mathrm{GBTP}_1(\{2,3^*\};2m+1,m\times(2m-3))$.

Lemma 8.4. *Suppose there exists a* $TD(5,n)$. *If* $0\le g\le n$ *and that there exists an* $IGBTP_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times 5)$ *for* $m\in\{n,g\}$, *then there exists an* $IGBTP_1(\{2,3^*\},2M+9,(M+4)\times(2M+5);9,4\times 5)$, *where* $M=4n+g$.

Proof. By Lemma 7.11, there exists an $\mathrm{FGBTD}(2,2^t)$ for all $t\in\{4,5\}$. By Proposition 6.9, there exists an $\mathrm{FGBTD}(2,(2n)^4(2g))$. Now apply Proposition 6.4 to obtain an $\mathrm{IGBTP}_1(\{2,3^*\},2M+9,(M+4)\times(2M+5);9,4\times 5)$. ∎

Lemma 8.5. *There exists an* $IGBTP_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times5)$ *for any* $m\ge 10$, *except for* $m\in\{16,20,24,28,32,36,46,50\}$.

Proof. Let $E=\{16,20,24,28,32,36,46,50\}$. An $\mathrm{IGBTP}_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times5)$ can be constructed for $10\le m\le 57$ and $m\notin E$, except for $m=51$. Details are provided in Table 8.2. When $m=51$, consider a TD(5, 11) and delete four points from a block to form a $\{4,5\}$-GDD of type $10^4 11$. Proposition 6.8 yields an $\mathrm{FGBTD}(2,20^4 22)$ and hence, Proposition 6.4 yields an $\mathrm{IGBTP}_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times5)$ with $m=51$.

We then prove the lemma by induction on $m\ge 57$. Let $E'=\{4n+g : n\in E, 10\le g\le 13\}$ and assume the lemma is true for $n<m$.

When $m\notin E'$, then write $m=4n+g$ with $13\le n<m$, $n\notin E$ and $g\in\{10,11,12,13\}$. Since a $\mathrm{TD}(5,n)$ which exists by Theorem 5.8, applying Lemma 8.4 with the corresponding n and g, we obtain the desired IGBTP.

When $m\in E'$, we have two cases.

- If $m=77$, the required IGBTP is given by Corollary 7.10.
- Otherwise, apply Lemma 8.4 with (n,g) taking values in $\{(15,14),(15,15),$ $(19,0),(18,18),(19,15),(23,0),(19,17),(22,18),(22,19),(27,0),(22,21),$ $(25,22),(25,23),(31,0),(25,25),(29,22),(29,23),(35,0),(29,25),(31,30),$ $(31,31),(39,0),(33,25),(39,38),(39,39),(49,0),(40,37),(42,42),(43,39),$ $(43,40),(43,41)\}$.

This completes the induction. ∎

Lemma 8.6. *A* $GBTP_1(\{2,3^*\},2m+1,m\times(2m-3))$ *exists for* $m\ge 4$, *except possibly for* $m\in\{12,13\}$.

Proof. A $\mathrm{GBTP}_1(\{2,3^*\};2m+1,m\times(2m-3))$ can be found via computer search for $4\le m\le 11$. The GBTPs are listed in [10].

For $m\in\{20,24,28,32,36,40,50,54\}$, set $M=m-5$ and apply Proposition 6.3 with $\mathrm{GBTP}_1(\{2,3^*\},11,5\times7)$ and the $\mathrm{IGBTP}_1(\{2,3^*\},2M+11,(M+5)\times(2M+7);11,5\times7)$ constructed in Corollary 7.10.

Finally, for $m\ge 14$ and $m\notin\{20,24,28,32,36,40,50,54\}$, set $M=m-4$ and apply Proposition 6.3 with $\mathrm{GBTP}_1(\{2,3^*\},9,4\times5)$ and the $\mathrm{IGBTP}_1(\{2,3^*\},2M+9,(M+4)\times(2M+5);9,4\times5)$ constructed in Lemma 8.5. ∎

Lemma 8.6 shows that a $\mathrm{GBTP}_1(\{2,3^*\},2m+1,m\times(2m-3))$ exists for all $m\ge 4$, except possibly for $m\in\{12.13\}$. Theorem 2.3(ii) now follows.

9. Conclusion. In this paper, we establish the first infinite families of equitable symbol weight codes, whose code lengths are greater than alphabet size and whose relative narrowband noise error-correcting capabilities tend to a positive constant as length grows. The construction method used is combinatorial and reveals interesting interplays equivalent combinatorial objects called generalized balanced tournament packings. These have enabled us to borrow ideas from combinatorial design theory to construct equitable symbol weight codes. In return, questions on equitable symbol weight codes offer new problems to combinatorial design theory. We expect this symbiosis to deepen.

Acknowledgement. The authors thank Charlie Colbourn for useful discussions.

REFERENCES

[1] R. J. R. Abel, C. J. Colbourn, and J. H. Dinitz, *Mutually orthogonal Latin squares (MOLS)*, in The CRC Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz, eds., CRC Press, Boca Raton, 2nd ed., 2007, pp. 160–193.

[2] M. Ardakani, G. Colavolpe, K. Dostert, H. C. Ferreira, D. Fertonani, T. G. Swart, A. M. Tonello, D. Umehara, and A. J. H. Vinck, *Digital transmission techniques*, in Power Line Commun. - Theory and Applicat. for Narrowband and Broadband Commun. over Power Lines, H.C. Ferreira, L. Lampe, J. Newbury, and T.G. Swart, eds., John Wiley & Sons, 2010, ch. 5, pp. 195–310.

[3] E. Biglieri, *Coding and modulation for a horrible channel*, IEEE Commun. Mag., 41 (2003), pp. 92–98.

[4] Marco Buratti and Anita Pasotti, *Combinatorial designs and the theorem of weil on multiplicative character sums*, Finite Fields and Their Applications, 15 (2009), pp. 332 – 344.

[5] Yanxun Chang, *The existence of resolvable BIBD with k even and $\lambda = 1$*, Discrete Mathematics, 218 (2000), pp. 9 – 23.

[6] Yanxun Chang and L. Ji, *Optimal (4up, 5, 1) optical orthogonal codes*, Journal of Combinatorial Designs, 12 (2004), pp. 346–361.

[7] Y. M. Chee, S. H. Dau, A. C. H. Ling, and S. Ling, *Linear size optimal q-ary constant-weight codes and constant-composition codes*, IEEE Trans. Inform. Theory, 56 (2010), pp. 140–151.

[8] Y. M. Chee, H.M. Kiah, A.C.H. Ling, and C.M. Wang, *Optimal equitable symbol weight codes for power line communications*, in ISIT 2012 – Proceedings of the 2012 IEEE International Symposium on Information Theory, 2012, pp. 671–675.

[9] Y. M. Chee, H.M. Kiah, P. Purkayastha, and C.M. Wang, *Importance of symbol equity in coded modulation for power line communications*, in ISIT 2012 – Proceedings of the 2012 IEEE International Symposium on Information Theory, 2012, pp. 666–670.

[10] Y. M. Chee, H. M. Kiah, A. C. H. Ling, and C. Wang, *Addendum to optimal equitable symbol weight for power line communications* `http://www1.spms.ntu.edu.sg/~kiah0001/optimalESWC.html`, Jan. 2013.

[11] Y. M. Chee, H. M. Kiah, and C. Wang, *Generalized balanced tournament designs with block size four.* Preprint, 2012.

[12] Y. M. Chee, A. C. H. Ling, S. Ling, and H. Shen, *The PBD-closure of constant-composition codes*, IEEE Trans. Inform. Theory, 53 (2007), pp. 2685–2692.

[13] W. Chu, C. J. Colbourn, and P. Dukes, *Constructions for permutation codes in powerline communications*, Des. Codes Cryptogr., 32 (2004), pp. 51–64.

[14] ———, *On constant composition codes*, Discrete Appl. Math., 154 (2006), pp. 912–929.

[15] C. J. Colbourn, T. Kløve, and A. C. H. Ling, *Permutation arrays for powerline communication and mutually orthogonal Latin squares*, IEEE Trans. Inform. Theory, 50 (2004), pp. 1289–1291.

[16] C. Ding and J. Yin, *Algebraic constructions of constant composition codes*, IEEE Trans. Inform. Theory, 51 (2005), pp. 1585–1589.

[17] ———, *Combinatorial constructions of optimal constant-composition codes*, IEEE Trans. Inform. Theory, 51 (2005), pp. 3671–3674.

[18] ———, *A construction of optimal constant composition codes*, Des. Codes Cryptogr., 40 (2006), pp. 157–165.

[19] C. Ding and J. Yuan, *A family of optimal constant-composition codes*, IEEE Trans. Inform. Theory, 51 (2005), pp. 3668–3671.

[20] P.J. Dukes, *Coding with injections*, Designs, Codes and Cryptography, (2011), pp. 1–10.

[21] D. Dzung, I. Berganza, and A. Sendin, *Evolution of powerline communications for smart distribution: from ripple control to OFDM*, in ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, 2011, pp. 474–478.

[22] P. Frankl and M. Deza, *On the maximum number of permutations with given maximal or minimal distance*, J. Comb. Theory Ser. A, 22 (1977), pp. 352–360.

[23] F. Gao and G. Ge, *Optimal ternary constant-composition codes of weight four and distance five*, IEEE Trans. Inform. Theory, 57 (2011), pp. 3742–3757.

[24] A. Haidine, B. Adebisi, A. Treytl, H. Pille, B. Honary, and A. Portnoy, *High-speed narrowband PLC in smart grid landscape – state-of-the-art*, in ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, 2011, pp. 468–473.

[25] S. Huczynska, *Powerline communication and the 36 officers problem*, Phil. Trans. R. Soc. A, 364 (2006), pp. 3199–3214.

[26] Sophie Huczynska, *Equidistant frequency permutation arrays and related constant composition codes*, Des. Codes Cryptogr., 54 (2010), pp. 109–120.

[27] S. Huczynska and G. L. Mullen, *Frequency permutation arrays*, J. Combin. Des., 14 (2006), pp. 463–478.

[28] Petteri Kaski and Patric R. J. Östergård, *Classification Algorithms for Codes and Designs*, no. 15 in Algorithms and Computation in Mathematics, Springer-Verlag, Berlin Heidelberg, 2006.

[29] E. R. Lamken, *Generalized balanced tournament designs*, Trans. Amer. Math. Soc., 318 (1990), pp. 473–490.

[30] ———, *Existence results for generalized balanced tournament designs with block size 3*, Des. Codes Cryptogr., 3 (1992), pp. 33–61.

[31] ———, *Constructions for generalized balanced tournament designs*, Discrete Math., 131 (1994), pp. 127–151.

[32] J. Lin, J. Chang, R. Chen, and T. Kløve, *Distance-preserving and distance-increasing mappings from ternary vectors to permutations*, IEEE Trans. Inform. Theory, 54 (2008), pp. 1334–1339.

[33] J. Liu, B. Zhao, L. Geng, Z. Yuan, and Y. Wang, *Communication performance of broadband PLC technologies for smart grid*, in ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, 2011, pp. 491–496.

[34] Y. Luo, F.-W. Fu, A. J. H. Vinck, and W. Chen, *On constant-composition codes over* Z_q, IEEE Trans. Inform. Theory, 49 (2003), pp. 3010–3016.

[35] N. Pavlidou, A. J. H. Vinck, J. Yazdani, and B. Honary, *Power line communications: state of the art and future trends*, IEEE Commun. Mag., 41 (2003), pp. 34–40.

[36] N. V. Semakov and V. A. Zinoviev, *Equidistant q-ary codes with maximal distance and resolvable balanced incomplete block designs*, Problemy Peredači Informacii, 4 (1968), pp. 3–10.

[37] D. J. J. Versfeld, A. J. H. Vinck, and H. C. Ferreira, *Reed-solomon coding to enhance the reliability of M-FSK in a power line environment*, in ISPLC 2005: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, 2005, pp. 100–104.

[38] D. J. J. Versfeld, A. J. H. Vinck, J. N. Ridley, and H. C. Ferreira, *Constructing coset codes with optimal same-symbol weight for detecting narrowband interference in M-FSK systems*, IEEE Trans. Inform. Theory, 56 (2010), pp. 6347–6353.

[39] A. J. H. Vinck, *Coded modulation for power line communications*, AEÜ - Int J. Electron. Commun., 54 (2000), pp. 45–49.

[40] Richard M. Wilson, *Cyclotomy and difference families in elementary abelian groups*, J. Number Theory, 4 (1972), pp. 17–47.

[41] R. M. Wilson, *An existence theory for pairwise balanced designs. I. Composition theorems and morphisms*, J. Combin. Theory Ser. A, 13 (1972), pp. 220–245.

[42] J. Yin, J. Yan, and C. Wang, *Generalized balanced tournament designs and related codes*, Des. Codes Cryptogr., 46 (2008), pp. 211–230.

[43] W. Zhang and L. Yang, *SC-FDMA for uplink smart meter transmission over low voltage power lines*, in ISPLC 2011: Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, 2011, pp. 497–502.